A NEW GEOMETRY

FOR HIGH SCHOOLS

A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.

FOREWORD BY

R. VAIDYANATHASWAMY, M.A., D.Sc., F.R.S.E.

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WITH A FOREWORD

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PREFACE.

THOUGH Geometry is an ancient science and numerous modern text-books have been put on the market, no apology is needed for writing a fresh text-book on the subject. The author's long and intimate acquaintance with Geometry as taught in High Schools and Colleges has shown him that the methods of teaching and presentation of Geometry have by no means become stereotyped and there is a large body of earnest and progressive teachers who welcome all contributions to improvement. The Reports of the various Teachers' Associations on the Teaching of Geometry are landmarks in the gradual evolution of Geometry to betterment. Special mention may be made of the Report on the Teaching of Geometry in Schools, prepared for the Mathematical Association in England by a small Sub-Committee of eminent mathematical teachers. This has been a source of inspiration to text-book writers and a few texts have recently appeared incorporating the recommendations in the Report and adapting them to the needs of the Public Schools in England. In the absence of a similar publication suitable for Indian Schools, the present author has ventured to write this work more or less on the basis of the suggestions of the Report.

This text-book in Geometry has been prepared with a view to meet the special needs of the Indian High Schools, especially those of Mysore and the Madras Presidency. The subject-matter closely follows the Madras S. S. L. C. Syllabus in Geometry for the C Group, and is presented in *four* books, arranged in chapters by topics.

Book I on 'the Straight Line' deals with angles, parallels, triangles, parallelograms, construction of rectilineal figures and loci.

The Introductory Chapter in Book I provides a preliminary course in the fundamental concepts and the use of mathematical instruments. The exercises set in this chapter are of a practical and experimental character and introduce the pupil informally to important geometrical facts, the proofs of which he will learn at the deductive stage.

Book II on 'Areas' deals with formulæ for the areas of parallelograms and triangles and gives illustrations and explanations of the geometrical theorems corresponding to the well-known elementary algebraic identities. The Theorem of Pythagoras with its extensions and the Apollonius' Theorem are also treated in this book. This book concludes with an introduction to plotting of points and graphs.

Book III on 'the Circle' deals with the properties and construction of circles and incidentally introduces the reader to interesting geometrical patterns and designs.

Book IV on 'Ratio and Proportion' includes a brief treatment of similar triangles, trigonometric ratios and their simple applications.

The complete work (Books I to IV) is more than adequate for the needs of High School pupils and may be used also in junior collegiate classes. The Intermediate Science students of the Mysore University who have a course of compulsory mathematics will find this book serviceable.

The following are the special features of the book :—

(i) In the sequence of theorems, the author has made a little departure which enables the related

propositions to come together in a logical order; for example, the four theorems on congruence which are usually scattered in different places, come here consecutively as Theorems 13, 14, 15 and 16.

Theorems and the constructions based on them are also grouped together.

(ii) Each proposition is given a very full and adequate treatment with corollaries, notes and worked examples for the guidance of the pupil. Practical applications of geometrical facts are pointed out to show that Geometry is not a futile science but an interesting and useful one serving the practical needs of man.

(iii) Historical anecdotes and sketches are provided, as occasions arise, to relieve the student from the monotony of abstract reasoning and to relate the subject to human interests. As it is very important that our students should appreciate the nature and extent of the ancient Hindu contribution to mathematics, especially to Geometry, the author has selected several typical examples, theorems and arguments from the texts of Aryabhata, Brahmagupta, Bhaskara and Mahaviracharya.

(iv) Important notes of educational value intended for teachers, and not directly useful for examination purposes, are printed in small type, while standard riders which should be studied more or less as book-work appear in bold and prominent type. All theorems and problems which are to be studied as book-work appear in italics.

(v) A carefully graded set of mostly original examples is given at the end of almost every proposition to impress on the student the scope of its theoretical and practical applications. Results of standard theorems occurring late in the text are

introduced to the student in advance as riders in the earlier chapters. This is with a view to enable the student to recognise them as old friends and get a better grasp over them when he enters upon their formal study later.

(vi) Revision papers arranged in sets of four questions in each paper are provided at the end of each book. Many recent Geometry papers of Madras and Mysore S. S. L. C. Examinations and the European High School Examinations are also added.

Answers and hints to the solution of difficult examples are given for each book.

In the preparation of this text, the following valuable works have been consulted :—

(1) The Thirteen Books of Euclid's Elements, by Sir Thomas Heath.

(2) The Teaching of Geometry, by David Eugene Smith.

(3) The Report on the Teaching of Geometry in Schools (already mentioned), 1925.

The author's thanks are due to his friends, Mr. R. Vaidyanathaswamy, M.A., D.SC., F.R.S.E., Reader, Madras University, and Mr. A. Narasinga Rao, M.A., L.T., of the Presidency College, Madras, for important suggestions and criticisms while the book was passing through the press.

The author should specially mention with gratitude Dr. R. Vaidyanathaswamy's kindness in finding time in the midst of many engagements to write a Foreword to this book.

MYSORE, *August*, 1928. A. A. KRISHNASWAMI AYYANGAR.

FOREWORD.

MR. A. A. KRISHNASWAMI AYYANGAR, M.A., L.T., is a teacher of 15 years' standing well-known as a keen worker in Geometry and an assiduous student of the history of ancient mathematics. A book written by one of his attainments may be expected to mark an advance on the current Geometrical text-books.

A casual glance over the pages of 'A New Geometry for High Schools' will reveal several interesting features. The most striking of these is perhaps the critical and anecdotal information and the numerous historical glimpses scattered throughout the book. It is often forgotten by teachers and text-book writers alike that, in teaching a professedly abstract science like mathematics, the suggestion of an historical background is all the more indispensable, as it furnishes the only possible means of effective human appeal to the learner. If it is a matter for complaint that the high school curriculum leaves no impress of permanent cultural value on the pupil's mind, does not the fault lie mainly in our lifeless methods of teaching? How many opportunities of insinuating the seeds of a broader vision into the young receptive mind are literally thrown away in the class-room! I have therefore great pleasure in congratulating the author on the inclusion of this welcome and much needed feature in his book and wish to express the hope that the 'New Geometry' may set

the fashion in this respect to future mathematical text-books.

Other features worthy of mention are—the proper emphasis on experimental work with solids in the initial stages, the full explanations of the elements and varieties of geometrical reasoning, both in Chapter I, Section (4), and at other places where necessity arises, and the rational ordering of the subject-matter by the grouping of related theorems. One is also glad to note the large and varied selection of exercises, some of the most interesting of which are due to the author himself. A discussion of the foundations underlying Geometry is usually omitted from books of the high school standard as it would have to be either unsatisfactory or beyond the capacity of the pupils. The author has however made several occasional references to these purely logical aspects; these will be of interest to the teacher and may well prove an incentive for him to acquaint himself with the philosophy of his subject.

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MADRAS, *August* 1928.

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SYMBOLS AND ABBREVIATIONS.

$=$	*for*	is equal to.	rt. $\angle$	*for*	right angle.
$\neq$	,,	is not equal to.	$\triangle$	,,	triangle.
$-$	,,	diminished by.	$\perp$	,,	is perpendicular to.
$\sim$	,,	difference between.	$\parallel$	,,	is parallel to.
$+$	,,	together with.	$\square$m	,,	parallelogram.
$>$	,,	is greater than.	$\equiv$	,,	is congruent to.
$\ngtr$	,,	is not greater than.	$\therefore$	,,	therefore.
$<$	,,	is less than.	$\odot$	,,	circle.
$\nless$	,,	is not less than.	$\bigcirc^{ce}$	,,	circumference.
$\angle$s	,,	angles.	$10''$	,,	ten inches.

$A\hat{B}C$ or $\angle ABC$ *for* angle ABC.

adj.	*for*	adjacent.	opp.	*for*	opposite.
alt.	,,	alternate.	parl.	,,	parallel.
chd.	,,	chord.	perp.	,,	perpendicular.
const.	,,	construction.	pt.	,,	point.
cor.	,,	corollary.	quad.	,,	quadrilateral.
corresp.	,,	corresponding.	rect.	,,	rectangle.
cms.	,,	centimetres.	reqd.	,,	required.
equil.	,,	equilateral.	resp.	,,	respectively.
ext.	*for*	external or exterior.	sq.	,,	square.
fig.	,,	figure.	sqq.	,,	squares.
ft.	,,	foot or feet.	st.	,,	straight.
hyp.	,,	hypotenuse.	Th.	,,	Theorem.
int.	,,	internal or interior.	trap.	,,	trapezium.
i. e.	,,	that is.	vert.	,,	vertically.
isos.	,,	isosceles.	*viz.*	,,	namely.
mid.	,,	middle.	*N.B.*	,,	Note well.

BOOK I.

THE STRAIGHT LINE.

CHAPTER I.

INTRODUCTORY.

§ 1. Historical Introduction.

The objects that we see around us have various shapes, different sizes and occupy certain positions in space. The science that deals in an exact manner with the shape, size, position and construction of figures, analysing them into simpler elements and deducing from these various relations and facts is known as Geometry.

'Geometry' is derived from the Greek word *Geometria*—*gē*, the earth and *metron*, a measure which suggests that it had its origin in measuring lands. The story goes that in Egypt periodical floods of the Nile swept away land-marks and the frequent changes in the course of the river altered the sizes of the fields on either bank and affected thereby the taxes due to the king from the owners of the fields. The Egyptians were therefore obliged to invent a kind of Geometry to enable them to restore these land-marks, to re-survey the lands and levy fresh taxes.

We may mention that in India, the performance of religious ceremonies at auspicious moments enjoined in the Vedas, demanded the construction of altars of peculiar shapes as in the Sulva-Sutras, and this necessitated the study of Geometry as an art by itself and also as an aid to Astronomy.

An interesting example of a historic altar-problem is the well-known problem of the duplication of the cube (*i.e.*, of finding an edge of a cube whose volume is twice that of a given cube) which led to many important mathematical investigations. Myth traces its origin to the Oracle at Delos which declared that the altar of Apollo (cubical in shape) should be doubled for the plague in the city to cease.

Though in various civilised countries of the world from the very earliest times, Geometry was studied as an art, it was destined for the Greeks, the most civilised European nation in early history, to study Geometry as a science, to search for general principles underlying the various geometrical facts and to find a rational explanation for them.

To Euclid, the great Greek mathematician of the third century B.C., we owe the first profound and systematic exposition of all the geometrical facts then known. His work styled 'The Elements of Geometry' is so excellent and such a great monument of logical reasoning that even to-day, after the lapse of more than 2000 years, attempts to improve on its plan are recognised to be nearly futile. The cultural and disciplinary value of the work was very well-known to the ancient Greek philosophers who used to post up on their school-doors : 'Let no one come to our school who has not first learned the Elements of Euclid.'

§ 2 Definitions and Fundamental Concepts.

SOLID, SURFACE, LINE, POINT.

A body occupying space (say, a log of wood) is called a **Solid** in Geometry.

(In Physics, 'Solid' has a different significance and refers to the nature of the material of which a body is composed).

It is said to possess three dimensions, *viz.*, length, breadth and thickness.

The amount of space occupied by a solid is called its **volume.**

A solid is separated from surrounding space by boundaries called **surfaces** which may be **curved or plane.**

A surface has no thickness.

Example. The surface of a ball is curved; the surface of a glass-pane or the top of a table is plane.

The test of a plane surface is that if **any** two points be taken on it, a straight rod connecting the points must lie completely on the surface.

The geometrical plane surface is a pure concept and can hardly be met with in nature; but a carpenter can make artificially a surface as plane as possible by means of his planing-machine.

The boundaries of a surface or a portion of it as well as the intersection of two surfaces are **lines.**

A line which lies evenly between its extremes is said to be **straight,** otherwise **curved.** It has length, but no breadth or thickness.

For example, the edge of a flat ruler is a straight line, but the rim of a coin is a curved line.

Two planes intersect in a straight line.

For example, when you fold a piece of paper carefully, the crease so obtained is a straight line.

There are surfaces on which straight lines can be drawn, though not in every direction; for example, the curved

surface of a cone or a cylinder. Such surfaces are called **ruled surfaces.**

There are others on which no straight line can be drawn; for instance, the surface of a round ball.

A plane surface enclosed by straight lines is called a **rectilineal** or **rectilinear figure.** (Fig. 1.)

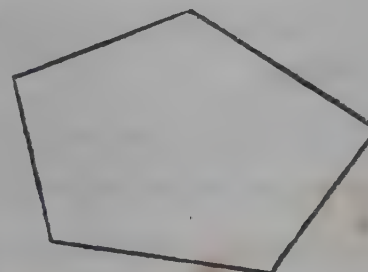


FIG. 1.

A plane figure enclosed by

	three	straight lines	is called	a **trilateral** or **triangle**;
,,	**four**	,,	,,	a **quadrilateral**;
,,	**five**	,,	,,	a **pentagon**;
,,	**six**	,,	,,	a **hexagon**;
,,	**seven**	,,	,,	a **heptagon**;
,,	**eight**	,,	,,	an **octagon**;
,,	**nine**	,,	,,	a **nonagon** or an **enneagon**;
,,	**ten**	,,	,,	a **decagon**;
,,	**n**	,,	,,	an **n-gon.**

A **polygon** is a plane figure enclosed by more than four straight lines.

The amount of surface enclosed by a figure is called its **area.** Figures having the same area are said to be **equal.**

The extremities of a line are **points.** So also are the intersections of two lines.

A **point** is that which marks a **position** and has no parts.

For example, the sharp end of a pin or a nail may be said to be a point.

(A geometrical line as well as a geometrical point like the geometrical surface is an ideal or a purely conceptual one. The so-called points that we mark with our pencils, however finely pointed, will be found to possess, when observed with a microscope, three dimensions, *viz.*, length, breadth and thickness.)

The following **facts** about points, straight lines and planes are fundamental and readily verified by our ordinary experience.

(1) *Two points determine a straight line.*

A straight line can be drawn from any one point to any other point. When we draw such a straight line, we are said to **join** the points.

(2) *Two straight lines cannot enclose a space.*

Two straight lines can intersect in only one point; for, if they meet in two points, they would either enclose a space or coincide with one another.

NOTE:—When the extremities of one straight line can be made to coincide with those of another, the two straight lines are said to be equal.

(3) *A straight line can be produced indefinitely to any length in the same straight line.*

(4) † *A straight line is the shortest distance between two given points.*

† This property of a straight line is not, strictly speaking, fundamental, being capable of proof.

Thus, if A, B, C be three points not in a straight line (Fig. 2), the distance between A and B along the straight line AB is less than the distance between the same two points along the broken lines AC, CB.

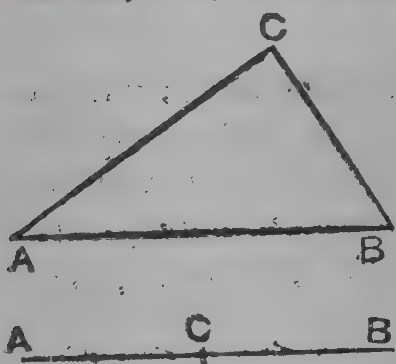


FIG. 2.

Def.:—If three points A, C, B lie on a straight line, they are said to be **collinear.**

If AC = CB, the straight line AB is said to be **bisected** at C and C is called the **middle point** of AB.

(5) *Three points not in a straight line determine a plane completely.*

In other words, **only one** plane can be made to pass through three points not in the same straight line.

(6) *If two points A, B of a straight line AB lie in a plane, then every point of the straight line AB and AB produced lies in the plane.*

It follows from (5) and (6) above that a plane can be determined uniquely either by

(i) a straight line and a point, or

(ii) two intersecting straight lines.

ANGLES.

If from a point A two straight lines AQ, AS be drawn, they are said to form an **angle**. A is called the **vertex** and AQ, AS the **arms** of the angle. AQ, AS are also said to **include** the angle QAS or SAQ or more

simply the angle A (when there is only one angle at A). (Fig. 3.)

If a straight line of whatever **length** starting from the initial position, AQ revolves about A as a hinge to the position AS in the plane QAS, it is said to **generate** the angle QAS. Thus, the **size of an angle** is **independent** of the **length** of the arms and **depends** only on the amount of **turning** or **rotation** done.

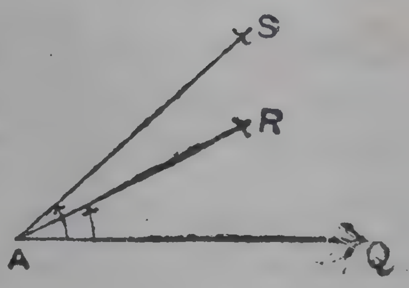


FIG. 3.

If AR be a position of the revolving line between AQ and AS, the angle QAR is said to be **less** than the angle QAS which is said to be **equal** to the sum of the angles QAR and RAS. The two angles QAR, RAS on opposite sides of the common arm AR are said to be **adjacent.**

If a straight line starting from AQ rotates about A in he plane QAS continuously in the same direction until it comes back to AQ, it is said to make a **complete revolution.**

Two angles are said to be **equal** when the arms of the one can be made to coincide with those of the other.

If in Fig. (3), the angle QAR be equal to the angle RAS, AR is called the **bisector** of the angle QAS.

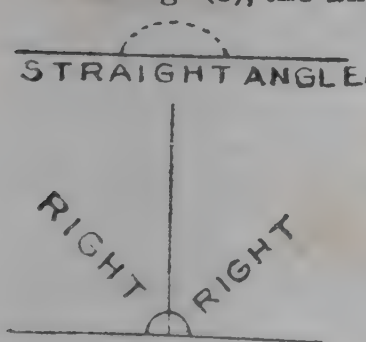


FIG. 4.

When the two arms of an angle are in the same straight line (Fig. 4), the angle is called a **straight angle.**

When one straight line stands on another straight line so as to

make the two adjacent angles equal (Fig. 4), each of these angles is called a **right angle** and the straight line standing on the other is said to be **perpendicular** to it.

A straight angle is equal to two right angles. A complete revolution is four right angles.

NOTE: From practical experience it is known that all right angles are equal. Therefore, we are justified in taking a right angle as the unit of angular measurement.

For convenience of measurement, a right angle is divided into 90 equal parts called degrees, each degree into 60 equal parts called minutes, and each minute into 60 equal parts called seconds.

Symbolically, 1 rt. $\angle = 90°$; $1° = 60'$; $1' = 60''$.

An **acute** angle is one which is less than a right angle. (Fig. 5.)

An **obtuse** angle is one which is greater than a right angle and less than two right angles. (Fig. 5).

An angle which is greater than two right angles and less than four right angles is called a **reflex angle.**

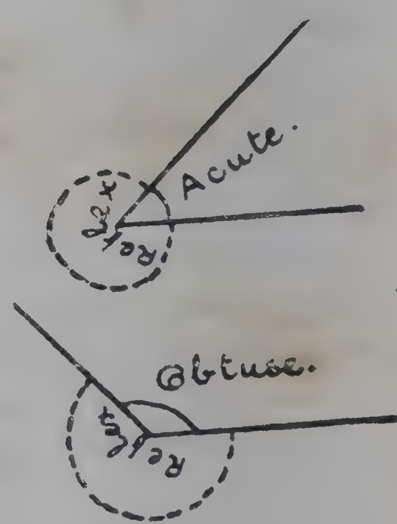


FIG. 5.

Thus, the angles marked with dots in Fig. 5 are reflex angles.

Two angles are said to be **complementary** or **supplementary** according as their sum is equal to **one** right angle or **two** right angles.

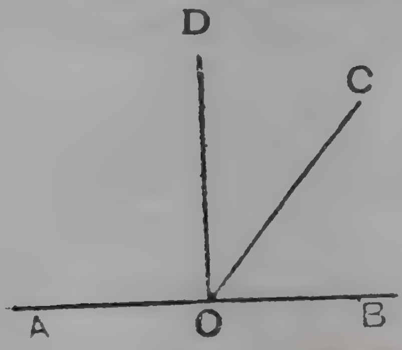
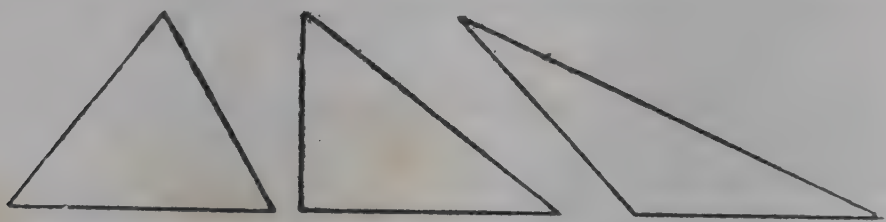


FIG. 6.

In Fig. (6), the angles BOC, COA are supplementary or supplements of each other; and the angles BOC, COD are complementary or complements of each other.

Triangles are called **acute-angled**, **right-angled** or **obtuse-angled** according as the greatest angle in them is **acute**, **right** or **obtuse**. (Fig. 7.)



Acute-angled Triangle. *Right-angled Triangle.* *Obtuse-angled Triangle.*

FIG 7.

Triangles when classified according to sides are named **equilateral**, **isosceles** or **scalene** (Fig. 8) according as **three** sides, **two** sides or **no two** sides are equal.

Isosceles Triangle. *Equilateral Triangle.* *Scalene Triangle.*

FIG. 8.

NOTE. If all the angles of a triangle are equal, the triangle is said to be **equiangular**. Such a triangle is always acute-angled.

In the same way, a **polygon** is said to be **equiangular** when all its angles are equal; it is said to be **equilateral** when all its sides are equal.

A **regular** polygon is one which is both equiangular and equilateral.

PARALLELS.

Two straight lines in the same plane which do not meet however far they are produced either way are said to be **parallel**. (Fig. 9.)

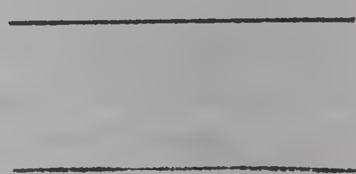


FIG. 9.

NOTE:—It is an important **fact** (ascertained by practical experience) that through a given point **only one** straight line can be drawn parallel to a given straight line. In other words, two intersecting straight lines cannot both be parallel to a third straight line. The whole theory of parallels is based on this principle.

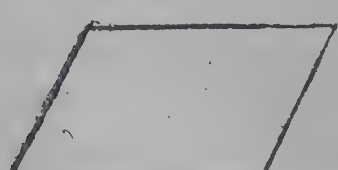


FIG. 10.

A quadrilateral whose opposite sides are parallel is called a **parallelogram.** (Fig. 10).

A **rectangle** is a parallelogram which has one angle a right angle. (Fig. 11.)

FIG. 11.

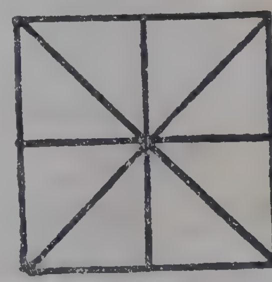


FIG. 12.

A **square** is a rectangle in which all the sides are equal. (Fig. 12).

CIRCLES.

A **circle** is a plane figure bounded by one line called the **circumference**, which is such that every point in it is at the same distance from a certain point within the figure; and this point is called the **centre** of the circle. (Fig. 13).

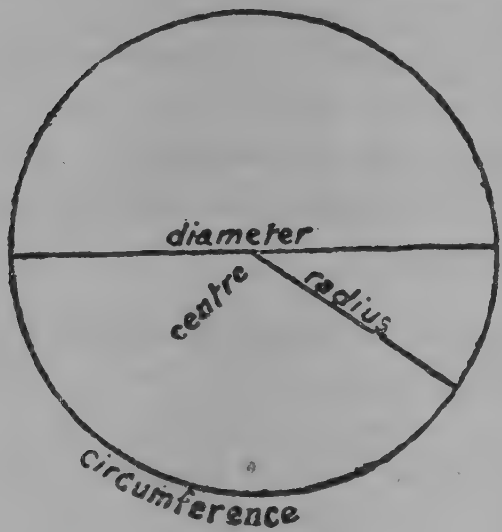


FIG. 13.

An **arc** of a circle is any part of the circumference.

A straight line drawn from the centre to the circumference is called a **radius** of the circle.

It follows from the definition that all radii of a circle are equal.

A straight line joining any two points on a circle is called a **chord**.

A **diameter** of a circle is the straight line drawn through the centre and terminated both ways by the circumference.

Since all radii of a circle are equal, all diameters are also equal.

If a circle is folded about a diameter, the two parts of the circle may be made to coincide. Each of these parts is called a **semi-circle**.

SYMMETRY.†

When the parts of a plane figure on each side of a straight line can, on being folded about the line, be brought into coincidence, (in other words, when the figure on one side of the line is the image of the figure on the other side), the figure is said to be **symmetrical** about the line and the line itself is called an **axis of symmetry** (Fig. 14.) This kind of axial symmetry is found in the letter T.

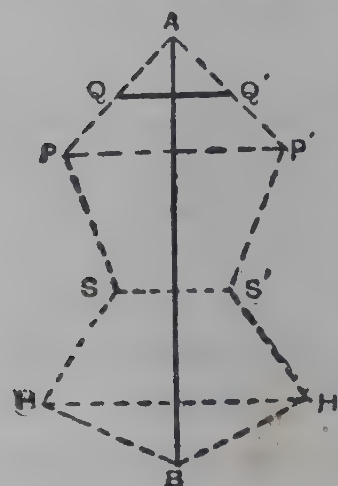


FIG. 14.

A circle is symmetrical about any diameter (Fig. 13); a square is symmetrical about (i) the line joining a pair of opposite vertices and (ii) the line joining the middle points of a pair of opposite sides. (Fig. 12).

Two figures or parts of the same figure in a plane may sometimes be made to coincide with each other, when each figure is rotated about a certain point in the plane through 180°. In this case the figures are said to be **symmetrically situated** with respect to the point, which is called the **centre of symmetry** (Fig. 15). Such a kind of central symmetry is found in the letter S.

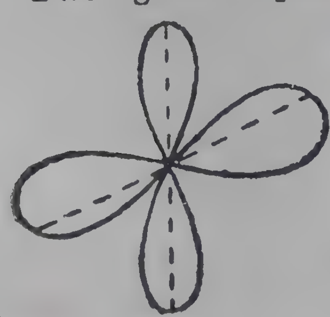


FIG. 15.

A circle, a square and a parallelogram are examples of plane figures which possess central symmetry.

† For a fuller discussion of 'Symmetry' *vide* Book III.

* SOLIDS.

The most familiar solid is a brick or a rectangular box. A solid of this shape is called a **rectangular block.** It has length, breadth and height.

The plane figures which bound a solid are called its **faces.** The straight lines at which two adjacent faces meet are called the **edges** of the solid. The corners or the points at which two adjacent edges meet are known as the **vertices** of the solid. The different planes meeting at a vertex are said to enclose what is called a **solid angle**† at that vertex. When a solid is cut by a plane, the new surface that is formed is called a **plane section** of the solid.

When two planes do not meet, however far they may be produced, they are said to be parallel.†

Solids bounded by plane surfaces are called **polyhedra.**

Regular solids are those whose solid angles are formed by the same number of equal regular polygons.

There are only **five** regular solids:

(i) the regular **tetrahedron** bounded by **four equal equilateral triangles**;

(ii) the regular **hexahedron** or **cube** bounded by **six equal squares**;

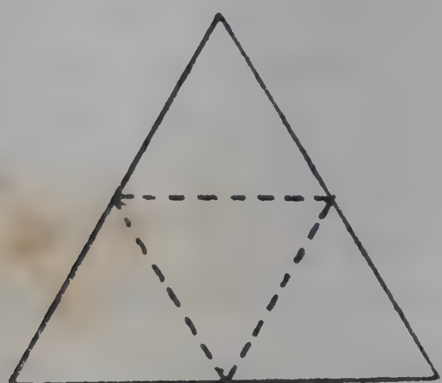
(iii) the regular **octahedron** bounded by **eight equal equilateral triangles**;

(iv) the regular **dodecahedron** bounded by **twelve equal regular pentagons**; and

† It will be instructive for boys to compare the definitions of 'solid angles' and 'parallel planes' respectively, with those of 'plane angles' and 'parallel lines'.

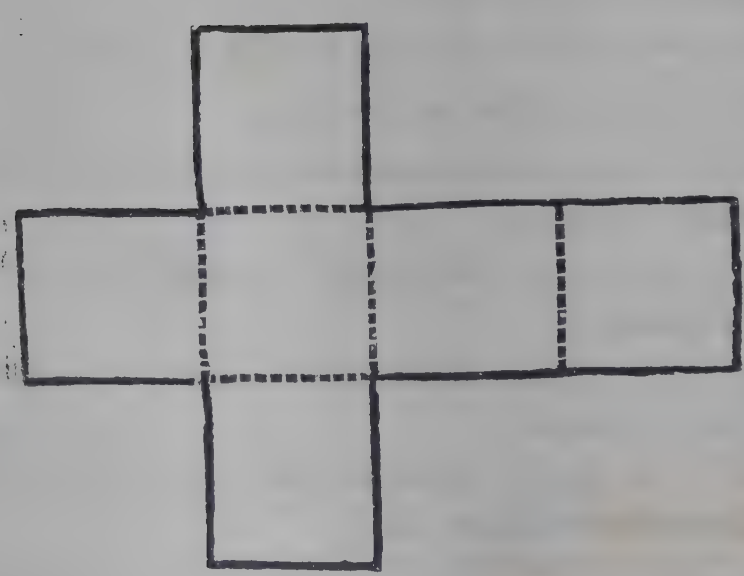
(v) the regular **icosahedron** bounded by **twenty equal equilateral triangles.**

These are also called Platonic bodies, because the School of Plato about 380 B.C. studied their properties systematically.



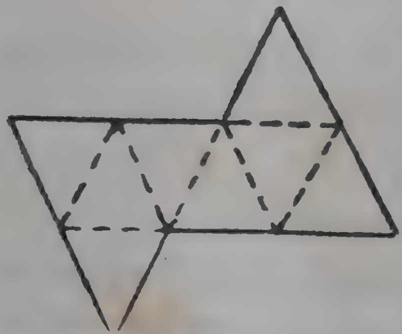
Net of a Regular Tetrahedron.

FIG. 16.



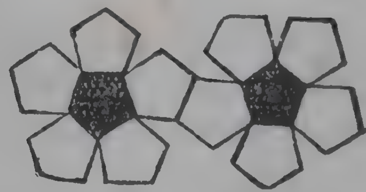
Net of a Cube.

FIG. 17.



Net of a Regular Octahedron

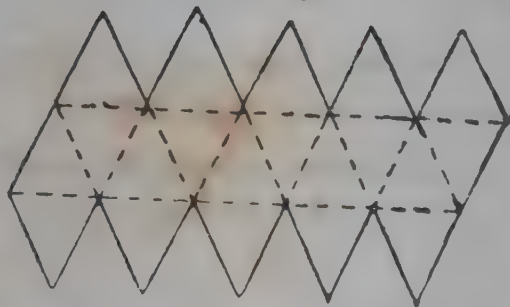
FIG. 18.



Net of a Regular Dodecahedron.

FIG 19.

It is easy to construct paper-models of the five regular solids by cutting a piece of paste-board into the figures called 'nets' (Figs. 16, 17, 18, 19, 20) and folding up the different parts into the required forms



Net of a Regular Icosahedron

FIG 20.

N B. The nets given in the figs. 16—20 can be easily drawn by superposing the successive faces of the model on paper and tracing their outlines.

A solid whose two ends are plane rectilinear figures parallel to each other and whose sides are rectangles or parallelograms is called a **prism** (Fig. 21.)

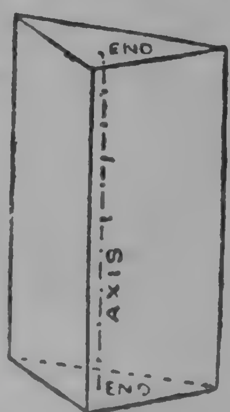


FIG. 21.

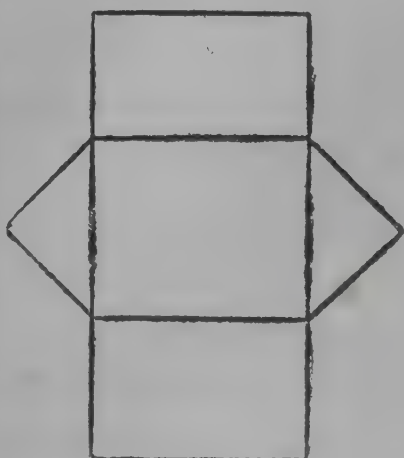


FIG. 22.

The net of a triangular prism is given in Fig. 22.

A pyramid is a solid standing on any rectilinear plane figure as base and having triangular sides terminating in a vertex above the base.

A tetrahedron is an example of a pyramid.

Besides the above, there are certain other kinds of simple solids called **solids of revolution** which are obtained by rotating a plane figure about some straight line in the plane as **axis.** These figures have no corresponding 'nets.'

A sphere is a solid obtained by rotating a semi-circle about its diameter as axis.

A cylinder is a solid obtained by rotating a rectangle about one of the sides as axis. (Fig 23).

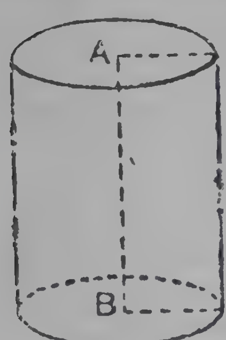


FIG. 23.

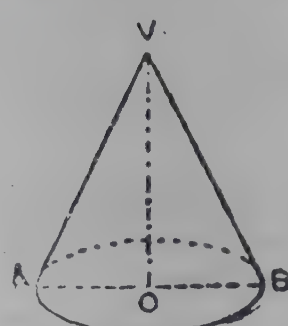


FIG. 24.

A cone is a solid obtained by rotating a right-angled triangle about one of the sides containing the right angle as axis. It is more correctly described a right circular cone, for its base is a circle and the axis is at right angles to it. (Fig. 24).

SYMMETRY OF SOLIDS.

Corresponding to the **two** kinds of symmetry (T and S) possible in plane figures, there are **three** kinds of symmetry that may be recognised in solids :

(i) Symmetry about a **plane.**
(ii) ,, **line.**
(iii) ,, **point.**

Like line-symmetry in a plane, the **symmetry** of a solid about a **plane** is the property that the figure on one side of the plane is the image of the figure on the other side.

For example, a sphere is symmetrical about **any plane** through the centre.

A solid is **symmetrical about a line** when every section of the solid by a plane passing through the line is symmetrical about the line.

For example, a right circular cone is symmetrical about its axis.

A solid is **symmetrical about a point** when every section of the solid by a plane passing through the point is symmetrical about the point.

For instance, a sphere is symmetrical about its centre.

NOTE TO THE TEACHER: The introduction of solids in the class-room will give geometry so much life and reality and not a little benefit can be derived by inducing the boys to prepare their own models out of very simple and inexpensive materials such as card-board, paper, wires, and strings. First-hand observation of the form and the properties of solids by actually handling them affords good training in the power of space-intuition and forms a necessary link between theoretical geometry and its practical applications. So it is desirable that the geometry of the common solids be taught early enough as part of the Practical Geometry course in our schools.

EXERCISE I.

1. How would you test whether (i) a surface is plane, (ii) a line is straight?

2. In what different ways can you get (i) a straight line, (ii) a point?

3. Draw three straight lines (i) so as to enclose a space, (ii) so as not to enclose a space.

4. Name some solids which are bounded purely by (i) triangles, (ii) quadrilaterals, (iii) pentagons.

5. The straight line joining any two non-adjacent vertices of a polygon is called a **diagonal.** How many diagonals are there in (i) a heptagon, (ii) a decagon?

Which is the plane rectilinear figure having no diagonals?

Which is the polygon having as many diagonals as there are sides?

Name the polygon which has thrice as many diagonals as there are sides.

6. Why does a three-legged stool stand firmly, while a four-legged one often does not?

7. Show that any convex path of broken lines between A and B within the triangle ABC (as in figure 25) is longer than AB but shorter than AC + CB. (*A path is said to be convex when the join of any two points in it is always on the same side of the path*).

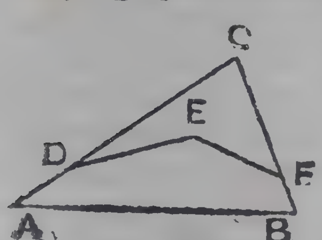


FIG. 25.

8. Are there straight lines which never meet however far they are produced and yet are not parallel? Point out some familiar examples of such lines.

9. Is there any figure which is not plane but bounded by a line, every point of which is equidistant from a given point?

10. (i) Give examples of solids in which (1) there are parallel faces, (2) no two faces are parallel, and (3) only two faces are parallel.

(ii) Give examples of two solids which are bounded by only two surfaces, one plane and the other curved.

(iii) Give examples of solids bounded by (1) four, (2) five, (3) six, and (4) seven plane faces and give descriptions of each face.

11 (*a*) What is the smallest number of plane faces that will enclose (i) a solid angle, (ii) a space?

(*b*) What is the greatest number of equal regular polygons that can enclose (i) a solid angle, (ii) a space?

(*c*) Show that *one* curved surface can enclose a space.

12. Are the following solids **prisms** or **pyramids**:—

(i) A cube, (ii) a tetrahedron, (iii) a wedge, (iv) a rectangular box, (v) a hexagonal pillar?

13. Draw a net of a rectangular block of any convenient dimensions on card-board, and show how you will use it to construct a solid model. How many rectangles are there in the net, and which of them are equal in pairs?

14. In as many different ways as you can, draw the net of (i) a prism on a quadrilateral base, (ii) a regular octahedron, (iii) a pyramid on a square base.

Note in your net the sides that should come together when the net is folded so as to take the shape of the required solid.

15. If S be the number of solid angles, F the number of faces, E the number of edges, tabulate the values of S, F and E for the following solids:—(1) Tetrahedron, (2) cube, (3) prism on a quadrilateral base, (4) octahedron, (5) icosahedron, (6) dodecahedron, (7) pyramid on a pentagonal base, and verify the property $F + S = E + 2$. (Euler's Theorem).

16. Is it possible to wrap a piece of paper over the following solids without folding or crumpling it:—A ball,

a cone, a garden-roller and a brick? Can you rule straight lines in (i) any, (ii) particular directions on the surfaces of these solids?

17. Are the following plane figures symmetrical about any line in the plane:—a semi-circle, a square, a rectangle, an equilateral triangle, a regular pentagon, a regular hexagon, a figure enclosed by two radii of a circle and the arc between them, and a pair of circles? In each case, indicate the straight line or lines about which there is symmetry.

18. Examine whether the following geometrical figures have central symmetry:—(1) A circle, (2) a square, (3) a regular pentagon, (4) a parallelogram, and (5) a regular hexagon.

19. (i) Indicate the axes of symmetry in the following block capital letters:—A, B, D, H, M, O, T, U, V, W, X, Y.

(ii) Which block capital letters have central symmetry?

20. (i) If a plane figure is symmetrical about a line AB show that every straight line perpendicular to AB which cuts and is terminated by the figure is bisected by AB. If this figure is revolved about AB as axis, what kinds of symmetry are possessed by the solid thus obtained?

(ii) If a plane figure is symmetrical about a point O, show that every straight line through the point which cuts and is terminated both ways by the figure is bisected at O.

Show that a similar property holds also for a solid figure symmetrical about a point.

§ 3. The Use of Instruments.

In the drawing of geometrical figures, we are permitted to use certain instruments, *viz.* (i) a flat ruler with one edge marked in inches and tenths of an inch and the other in centimetres and millimetres; (ii) two set-squares, one with angles 30°, 60°, and 90° and the other with

angles 45°, 45° and 90°: (iii) a pair of dividers; (iv) a pair of compasses with a hard pencil point, and (v) a protractor.

In addition to these, we should possess pencils and compass leads with a fine **chisel edge.**

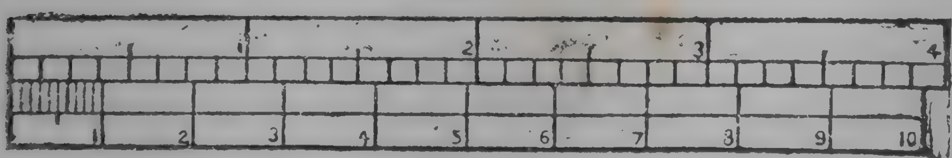
Points. A point should always be marked as an intersection of two short strokes, thus

P ✕ and never by a heavy dot thus —●— P.

Straight Lines. Use a hard pencil with a chisel edge to draw straight lines. When joining two points by a ruler, keep the ruler as close to (and not on) the two points as possible so that if the pencil-edge be placed at either of the points, it also touches the ruler; then starting from one of the points, draw the pencil-edge straight along the ruler towards the other point in a single movement.

Measurement of lengths. To measure lengths, we

inches.



centimetres.

FIG. 26.

generally use a graduated scale about 6″ long showing tenths of an inch on one side, centimetres and millimetres on the other and the graduation-marks must be brought as close to the line as possible. It is preferable for this purpose to have a scale with the graduated edge thin and bevelled or sloped off.

Set-squares are used for making angles of 30°, 45°, 60°, and 90° and for drawing perpendiculars and parallels through a given point to a given straight line. To make these angles, place the set-square containing the required angle flat on your paper and rule along the arms of the angle, but not quite close to the vertex of the angle. Remove the set-square and close up the angle by producing the arms to meet. Generally, the protractor is more convenient to make these angles.

To draw a perpendicular to a given straight line AB at a given point in it.

Place a set-square in position (1) such that a side containing the right angle may fall on AB. Holding a ruler firmly along the side opposite to the right angle, slide the set-square along the line of contact until the edge, say PQ perpendicular to AB passes through the given point, and rule the line P′Q′ along this edge in position (2).

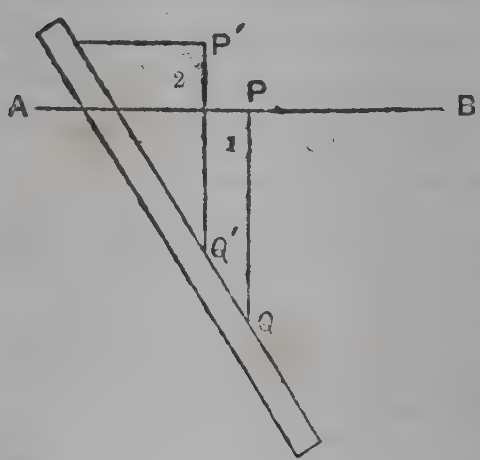


FIG. 27.

Similarly, we can draw straight lines through the given point to make with AB angles of 30°, 45°, 60° and their supplements.

To draw a perpendicular to AB from a point outside it.

Place a set-square with one of the sides containing the right angle on AB and slide it along AB until the perpendicular edge of the set-square is just close to the given point, say P. Rule through P along this edge. Then use a ruler or an edge of the set-square itself to produce this perpendicular until it cuts AB.

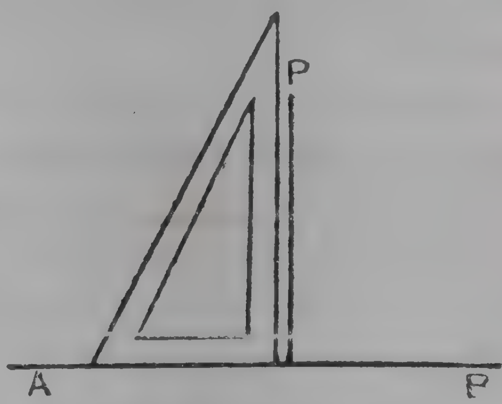


FIG. 28.

To draw a parallel to AB through a given point outside it.

Place a set-square with one side on AB. Place a straight edge (or another set-square with its longest side) along one of the other sides of the first set-square and holding the straight edge (or the second set-square) firmly, slide the first set-square along the line of contact so that the side which was originally in contact with AB moves and catches up the given point above or below AB, and rule along this side. Thus, in the figure PC, TS are parallel to AB.

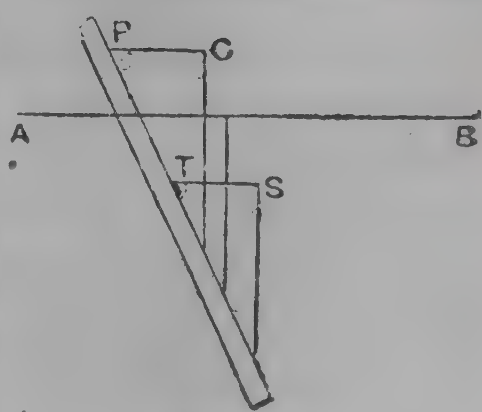


FIG. 29.

Caution: In ruling along a straight edge or an edge of set-square, you should neither *begin* from either extremity nor *end* there; for your pencil is likely to slip at these ends and the line which is intended to be straight will get crooked.

Dividers are used for comparing lengths, carrying distances and taking measurements.

To take a given distance on the dividers, open them out and adjust the two ends so as to fall on two graduation-marks of a scale with the given distance between them. If the angle between the arms of the dividers be kept steady, this distance between the ends may be carried anywhere and stepped off on any straight line.

To measure a given length, say AB, the dividers should be extended and the two ends adjusted so as to fall on A and B respectively; this distance may then be transferred to a scale such that one point of the dividers lies on a convenient mark of the scale and the other in its appropriate position in the scale. The distance may then be read off.

A pair of compasses is used to draw a circle or an arc of a circle having a given point as centre and passing through another given point. It is sometimes used also as a pair of dividers to transfer and step off distances.

Caution: 1. All pointed instruments like a pair of dividers should be used delicately, without injuring either the points of the instruments or the paper with which the points come in contact.

2. Take care to hold a pair of compasses or dividers by the head only or you may alter the distances between the end-points.

Measurement of angles: A semi-circular or rectangular protractor (*vide* Fig. 30, 31) is the instrument by means of which an angle is measured.

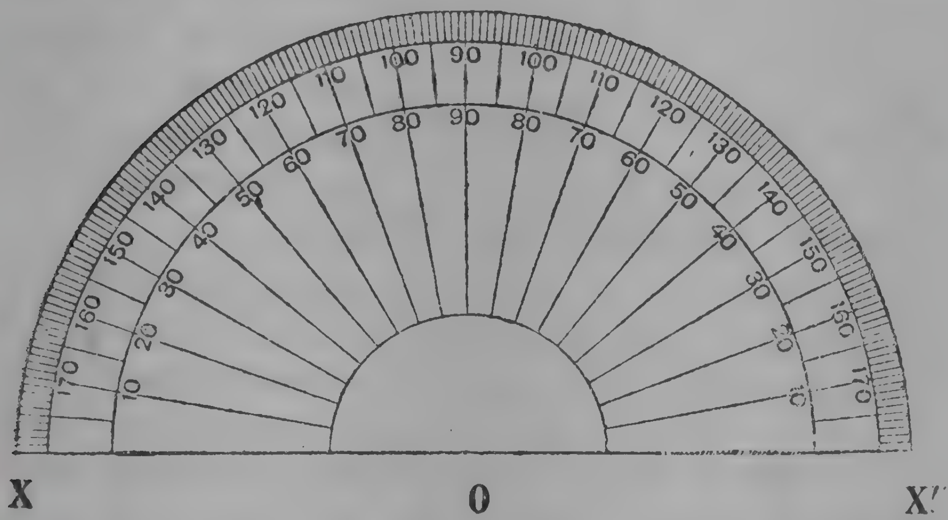


FIG. 30.

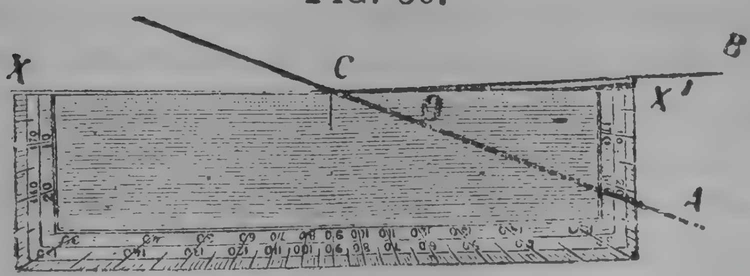


FIG. 31.

The rim of a protractor bears graduation-marks for every degree and these marks are numbered for every 10° in *two directions.* The numbering may be either continuous from end to end as in Fig 31 or symmetrical about the central mark (90) as in Fig. 30. The angle corresponding to any mark, say 20°, is that between the straight line joining it to the centre O of the base and the base line XOX′.

To measure any angle, say ACB (Fig. 32), adjust the position of the protractor so that the centre O of the base may fall on C and the base along CA and read the proper graduation (approximately if necessary) underneath which CB passes.

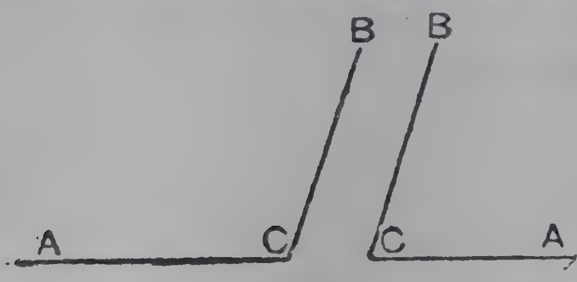


FIG. 32.

The usual method of drawing an angle of given magnitude is to draw a straight line CA to represent one arm of the angle, and place the protractor so that the centre O of its base may fall on C and the base line along CA. Then mark the point B just underneath the graduation representing the given magnitude and join CB. ACB is the required angle. The proper way of using the protractor is as follows: Draw a straight line CA as before and place the protractor so that the central point O of its base may fall on C and the graduation-mark representing the given magnitude on CA and rule CB along the base of the protractor. ACB is the required angle (*vide* Fig. 31).

NOTE. The double numbering on the rim of a protractor is a source of confusion to many pupils. For instance, the measure of an angle of 70° may be mis-read as 110°, while to draw an angle of 110°, the pupils generally draw an angle of 70°. The confusion is best cleared if the boys get into the habit of checking the measurements by their own *a priori* judgments or estimates about the nature of an angle, *i.e.*, whether it is acute, right, or obtuse.

POINTS OF THE COMPASS.

The four principal directions are North, East, South, and West. They are also called the cardinal points. Between

each pair of these principal directions taken in cyclic order, there are *seven* other directions named as in Fig. 33. Thus, in the figure there are thirty-two directions (or *points* in nautical language) and the angle between any two consecutive directions is $11\frac{1}{4}°$ or one-eighth of a right angle.

FIG. 33.

The abbreviations for North, East, South and West are N. E. S. W respectively. Sometimes, a direction midway between two primary directions is named by a compound of the names of the two directions; thus South-East (S.E.) means the direction between South and East, and South-South-East (S. S. E.) indicates the direction midway between South and South-East.

The phrase W. by N. should be construed as denoting the direction which is $11\frac{1}{4}°$ North of West, *i.e.*, midway between the directions W. and W. N. W. This direction may also be called W. $11\frac{1}{4}°$ N.

Similar phrases may be construed in the same way.

BEARING.

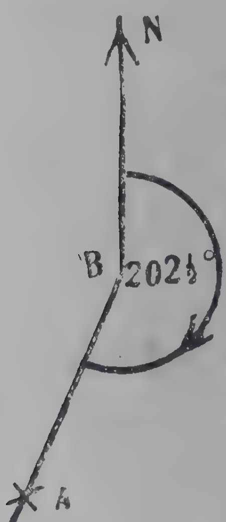


FIG. 34.

The **bearing** of an object A as seen from a place B is the number of degrees measured in an **easterly** direction between the **North line** at B and the line BA. If the angle is measured from the **geographical meridian** at B, the bearing is called a **true bearing** (or an azimuth) and if measured from the **magnetic North** (indicated by the magnetic needle) it is called a **magnetic bearing.**

Example. A bearing of $202\frac{1}{2}°$ is the same as the direction S. S. W. (*vide* Fig 34).

ANGLES OF ELEVATION AND DEPRESSION.

The line joining any two points at the same level is called a **horizontal line.**

Any line in the direction of the plumb-line or a freely falling body is called a **vertical line.**

A vertical line is easily seen to be perpendicular to a horizontal line.

The plane through any three points at the same level and not in the same straight line is called a **horizontal plane.** All lines in the horizontal plane are evidently horizontal lines; thus horizontal lines can be in different directions.

All lines perpendicular to the horizontal plane are vertical lines and all of them point to the same direction and are parallel to one another.

Any plane passing through a vertical line is called a **vertical plane.** But all lines in a vertical plane are not vertical lines.

Let A be the position of an observer looking at an object B not at the same level as A but higher or lower than A. Then the angle which AB makes with the horizontal line through A in the same vertical plane as AB is called an **angle of elevation** or **depression** according as B is above or below the level of A (*vide* Fig. 35.)

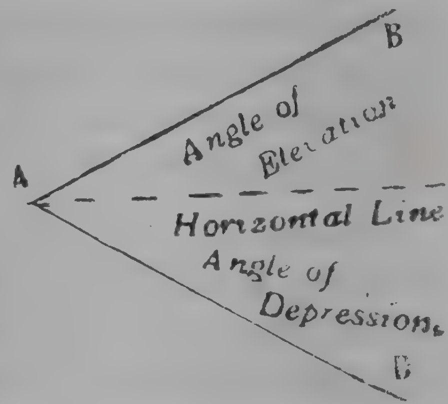


FIG. 35.

If the object B be at a considerable height above A, such as a cloud or a star, the angle of elevation is also called the **altitude** of the object.

DRAWING TO SCALE.

Whenever we have to prepare a plan of a building, a map of a piece of ground, etc., it is obviously out of the question to draw the figure to the actual dimensions. So, we adopt a convenient scale for our drawing, say, a line of certain length to represent a given actual length and take all the lengths in our figure proportionately. But it is important to remember that it will be absurd to use any scale for the angles and that all the *angles* in our representative figure

are exactly *equal* to their counterparts in the original. The most remarkable and fundamental fact to note in this connection is that if we draw any figure, big or small, which has the same angles as the original at the corresponding places, all the lengths of our figure will represent their originals in the same definite scale which we may call **the scale** of our figure. This fact helps us to deduce the properties of the original from those of its representative, the latter being more handy to study. For example, if two lengths are found to be equal in the representation, we can immediately infer that the corresponding lengths in the original are also equal.

Example 1. A, B, C are three islands. The direction of B from A is 10° W. of S., of C from B is 20° N. of E., and of A from C is 40° N. of W. Draw a plan to show the relative positions of the islands.

Take any point A and mark the cardinal directions at A. Draw the line AB in the direction 10° W. of S. Since the bearing of A from C is 40° N. of W. the bearing of C from A is 40° S. of E. and hence draw AC in this direction.

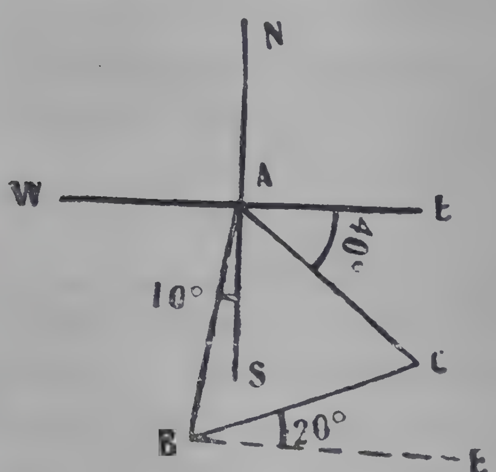


FIG. 36.

Again, draw the East line at B and BC in the direction 20° N. of E. to cut AC at C.

The vertices of the triangle ABC give the relative positions of the three islands.

NOTE. The direction opposite to any given direction is obtained easily by changing N. to S. and E. to W. and *vice-versa.*

Thus, the direction opposite to 40° N of W is the direction 40° S. of E. and the direction opposite to S. S. E is N. N. W.

Example 2. The angle of elevation of the top of a tree on a bank of a river as observed by a man at the opposite bank is 45°. Verify by drawing a plan that the height of the tree is equal to the breadth of the river.

Draw a straight line MF to represent the breadth of the river and at F erect a perpendicular FT to represent the tree. If M be the position of the observer, at M make the angle FMT = 45° the given angle of elevation, by means of a protractor.

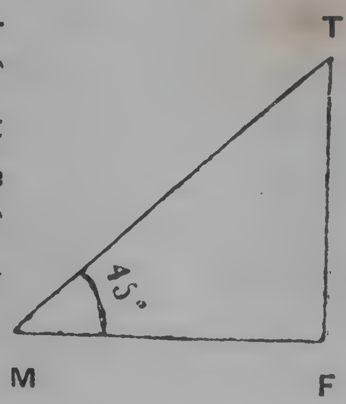


FIG. 37.

Then FT represents the height of the tree.

By measurement, FT = MF.

EXERCISE II.

1. Measure in inches, and also in centimetres:
   (i) the length of the paper you are using;
   (ii) the breadth of your table;
   (iii) the height of your room window;
   (iv) the length of your pencil.

2. Draw a straight line 10″ long and measure it in centimetres.

If your ruler is too short to draw the required line, how will you get over the difficulty?

3. A man walks 1 mile due North and then successively 1 mile due East, 1 mile due South and 1 mile due West. How far is he from his starting point? Represent his course in a diagram by taking 1 inch to stand for 1 mile.

NOTE: Whenever we cannot draw a figure to the stated dimensions, we use a scale with a convenient unit to represent a certain actual distance. For example, in a map of a country a length of 1 inch may represent an actual distance of 100 miles. This is expressed briefly: 1 in. = 100 miles.

4. Draw a long straight line and guess its middle point. Verify your guess by measurement.

5. Draw a long line AB. Measure off with your dividers equal lengths AC, CD,......and BC′, C′D′......... from each end of the line.

Show that the middle point of AB is the same as the middle point of CC′, DD′, etc.

6. Guess the lengths of your fingers and verify them by measurement.

7. Draw ten straight lines of different lengths and tabulate in parallel columns the guessed and the measured distances. Note the percentage of error in your guess in each case.

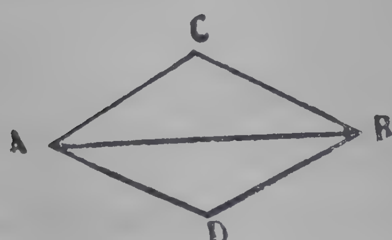


FIG. 38.

8. Draw a straight line AB 5 inches long and take two points C, D on opposite sides of AB. Join them to A and B as in the figure.

Draw another straight line

A′B′ of the same length and at A′ and B′ make angles 120° on both sides of the straight line as shown in the figure

Do the straight lines AB, A′B′ look equal ?

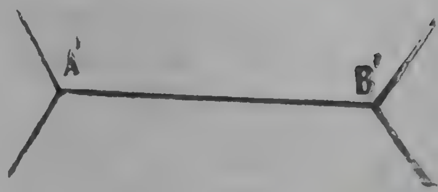


FIG. 39.

9. Draw any triangle on your paper and judge by the eye (1) which angle, and (2) which side is greatest ? Measure the angles and verify that their sum is equal to two right angles.

10. Draw a straight line BC, 8 cm. long and at B and C make angles ABC, DCB each equal to 70° on the same side of BC.

Examine whether BA and CD meet and if they meet, measure the angle at their point of intersection and the distances of the point from B and C.

Repeat the above construction by varying the angle.

Try the angles 20°, 30°, 80°, 90°, 100° and 150°.

Do your results suggest any general conclusions ?

11. Instead of making both the equal angles on the same side of BC as in Ex. 10, make one of them on one side and the other on the other side and examine whether AB and CD meet.

Repeat the construction with different angles and draw some general conclusions.

12. Verify the following methods for bisecting a given straight line, by means of a set-square :

(i) Let AB be the given straight line. At A and B on the same side of AB make angles BAC and ABC each equal to 30°. At C make the angle ACD equal to 60° and let CD cut AB at D. Then D is the middle point of AB.

(ii) In the above construction, instead of making angles 30° and 60° make angles of 45°. Still D will be the middle point of AB.

13. Draw two parallel straight lines AB, CD. On AB take three points X, Y, Z such that XY = YZ. Take any point O in the plane of the parallels. Join OX, OY, OZ and let them cut CD at X', Y', Z' respectively.

Verify $X'Y' = Y'Z'$.

Use the above construction to bisect any segment, say EF on CD.

14. Show how you will fold a piece of paper so as to get two straight lines at right angles. Bisect these angles again by folding. By repeated folding, get all the 32 points of the compass. How many times have you got to fold to get the thirty-two angles?

15. How will you bisect an angle by means of the protractor?

16. State whether the following angles are acute, right, or obtuse:

(i) 100°, its supplement, twice the supplement;

(ii) two complementary angles;

(iii) the angles of a parallelogram, when one of the angles is not a right angle;

(iv) the angles of a set-square;

(v) the plane angles at the vertices of a cube, a regular icosahedron, and a triangular prism.

17. Draw any convex pentagon. Measure the five interior angles and find their sum.

Do you get the same sum for any other convex pentagon also?

18. Draw any hexagon. Join one of its vertices to the remaining vertices of the figure excepting the two adjacent ones. How many triangles are formed? What is the sum of the angles in each triangle? What is the sum of all the interior angles of the hexagon? Is the angle-sum constant for all hexagons?

19. Draw any quadrilateral and its diagonals. Guess the sizes of all the angles formed. Verify by measurement.

20. Describe a circle of radius 2·5 inches. At the centre make a set of consecutive angles each equal to 60°.

How many such angles can be made?

Join the points (consecutively) where the arms cut the circle.

Examine whether (1) all the sides and (2) all the angles of the figure thus formed are equal. Compare a side of the hexagon with a radius.

Repeat the above construction for circles of varying magnitudes.

21. Draw a circle of radius 10 cms. and at the centre make five angles of 72° in succession. Verify that the arms of these angles cut the circle at the vertices of a regular pentagon.

Def.:—A polygon whose vertices lie on a circle is said to be inscribed in the circle.

22. Inscribe a square within a circle of diameter 12·6 centimetres.

23. Construct a triangle ABC given AB = 6·4 centimetres, BC = 8·9 centimetres, angle B = 100°.

Construct other triangles by varying the magnitudes of AB, BC and angle B.

24. Draw a straight line BC = 5 centimetres and at B make an angle CBX = 50°. From C draw CD perpendicular to BX. Measure CD.

With C as centre, and radii equal to 2, 3, 4, 5, 6, 7 centimetres draw circles and note which of them cuts BX or BX produced in (i) no points (ii) two points on the same side of B and (iii) two points on opposite sides of B.

Is there any case which does not come under the above heads?

25. A, B are two given points. Draw a circle with A as centre and radius equal to 7 centimetres and another circle with B as centre and radius equal to 10 centimetres.

Examine whether these circles have (i) no common point (ii) one common point or (iii) two common points when the distance AB varies from 2 centimetres to 20 centimetres; in the first case, note whether the circles lie, one within the other or each outside the other. Tabulate your results thus:

AB	Number of common points	Illustrative diagrams.
2 and less than 3		
3		
greater than 3 and less than 17		
17		
greater than 17; for example 20		

26. Construct triangles to the following measurements and in case the construction is impossible with the given data, explain at what stage it fails:

(i) AB = 7·3 cm., BC = 8 cm., CA = 9 cm., 8 cm. or 16 cm.

(ii) $AB = 7{\cdot}3$ cm, $BC = 8$ cm., $\hat{A} = 90°, 80°$ or $70°$.

(iii) $AB = 5$ in., $\hat{A} = 65°$, $\hat{C} = 50°, 65°$ or $120°$.

(iv) $BC = 3{\cdot}4$ in., $\hat{A} = 98°$, $AB = 1{\cdot}2$ in.

(v) $CA = 6{\cdot}8$ cm., $\hat{B} = 45°$, $AB = 3{\cdot}4$ cm.

(vi) $\hat{A} = 75°$, $BC = 3{\cdot}6$ in., $CA = 4$ in.

27. Construct quadrilaterals ABCD to satisfy the following data:—

(i) $AB = BC = CD = DA = 1$ in. $\hat{B} = 90°$ or $100°$.

(ii) $AB = CD = 3$ in., $BC = AD = 4$ in., $B\hat{C}D = 100°$ or $90°$.

(iii) $AB = AD = 5{\cdot}2$ cm., $BC = CD = 6$ cm., and $AC = 7{\cdot}2$ cm.

(iv) $AB = 3{\cdot}2$ cm., $BC = 4$ cm., $CD = 5$ cm., $AC = 3$ cm., $BD = 6$ cm.

(v) $AB = 1{\cdot}8$ in., $BC = 4$ in., $CD = 3$ in., $A\hat{B}C = 73°$, $B\hat{C}D = 57°$.

(vi) $C\hat{B}A = 130°$, $B\hat{A}D = 100°$, $B\hat{C}D = 73°$, $CD = 8$ cm., $DA = 6$ cm.

(vii) $BA = AD = DC = 1$ in., $A\hat{B}C = 60°$ and $B\hat{A}D = 120°$.

In each of the above cases, describe the shape of the figure obtained.

NOTE: To know where to begin, it is necessary to make a rough sketch of a quadrilateral and mark there-in the given data. This will suggest which triangle can be constructed first, and then the figure can be completed.

To construct a quadrilateral, we require *five* measurements which may belong to

either (i) four sides and one angle,

or (ii) three sides and two angles.

or (iii) two sides and three angles.

28. Is it possible to construct a quadrilateral with one side and any four angles?

29. Draw any parallelogram and its diagonals. Verify that the diagonals bisect each other.

Use this property of the diagonals,

(i) to bisect a given finite straight line,

(ii) to construct a parallelogram, given one side and two diagonals.

30. With the help of your set-squares, construct

(i) an equilateral triangle,

(ii) a square,

(iii) a regular hexagon,

(iv) a regular octagon

on a side of length 2 inches.

N.B. Note that the interior angles of the Figs. (i) to (iv) measure 60°, 90°, 120°, and 135° respectively.

31. Show how you would make the figures in Ex. 30 by paper-folding.

32. Show how you will use tracing paper to copy (i) a given angle (ii) a rectilinear figure.

33. To copy a triangle ABC, we may proceed as follows: Draw A′B′ equal to AB and with A′, B′ as centres and radii equal to AC, BC respectively draw arcs to cut at C′. Join C′A′, C′B′. Then A′B′C′ will be a copy of the triangle ABC.

To copy any given rectilinear figure, first divide the figure into triangles by joining one of the vertices to the remaining ones as in Ex. 18 and copy these triangles.

Use the above method to copy Fig. 40.

FIG. 40.

34. Draw a figure to represent the portion of a street 40 ft. wide watered by a man who stands at a point on one edge of the street with a hose-pipe throwing water to a distance of 50 feet. (Scale 20 feet to 1 inch.).

35. A man standing on a tank-bund sees a tree on the opposite bund in the direction 30° N of E. He walks along the tank-bund which is in the direction 15° W of N, a distance of $\frac{1}{2}$ furlong and finds the tree now due East of him.

Find the distance between the opposite bunds of the tank. Show that in both the positions the tree is at the same distance from the man.

36. Two men starting from the same point walk at the same speed in two different directions, one in the direction N. W by N and the other in the direction W. S. W. In what direction will each appear to move when seen by the other?

37. From the top of a tower, the angle of depression of a bridge is 30°. If the height of the tower is 200 feet, how far away is the bridge?

38. At two points 100 feet apart on opposite sides of a cocoanut tree the angles of elevation of its top are 50° and 60° respectively. Find the height of the tree.

39. The angle of elevation of a hill is observed by a man to be 70° He walks 200 yards towards the hill and finds the angle of elevation 85°. Find the height of the hill.

40. On a certain moonlight night a telegraph post 40 ft. high casts a shadow 15 ft. long. What is the altitude of the moon?

41. From the top of a light-house 300 feet high, I observe the angle of depression of a ship at anchor to be 30°. After descending 50 feet from the top, I find the angle of depression 20°. How far off is the ship from the light-house?

42. A vertical stick 7 feet long casts a shadow 10 feet long. If the stick be turned about its lower end in the vertical plane containing the shadow, find for what inclination of the stick to the horizontal the shadow will again be 10 ft.

43. The angles of elevation of the top of a tower from the top and bottom of a smaller one of height 40 feet are respectively 10° and 20°. Find the height of the former tower and the distance between the feet of the two towers.

44. Express as bearings the thirty-two points of the compass.

45. Find the bearings of the corners of a regular hexagonal field from its centre, assuming that one of the sides runs due North.

46. The bearings of a well from three houses equidistant from it are 190°, 340°, 80°. Draw a plan to indicate the relative positions of the houses and the well and read from your figure the bearings of each house as observed from the other two houses.

§ 4. (1) Foundations of Geometry—Axioms and Postulates.

We have already seen (§ 1) that the Science of Geometry deals with the shape, size, position and construction of figures, **analysing** them into simpler elements and **deducing** from these various relations and facts. Geometry is a

deductive science *i.e.* a science in which some things are taken for granted and others deduced therefrom. Obviously, everything cannot be proved and we have to start with some things admitting them to be incapable of proof. In the first place, we have principles common to all mathematics called **axioms** or **common notions.** The following are a few of them:

(*i*) *Things which are equal to the same thing are equal to one another.*

(*ii*) *If equals are added to equals, the sums are equal.*

(*iii*) *If equals are taken from equals the remainders are equal.*

(*iv*) *If equals be multiplied by equals, the products are equal.*

(*v*) *If equals be divided by equals, the quotients are equal.*

(*vi*) *The character of an inequality remains unaltered either by the addition (or subtraction), of equals; or by the multiplication (or division) by equal positive quantities.*

Example. If $A > B$, then $A + C > B + C$, $A - C > B - C$, $AC > BC$ and $\frac{A}{C} > \frac{B}{C}$ (C being a positive quantity.)

(*vii*) *The whole is greater than its part.*

Secondly, there are principles which are peculiar to the subject-matter of Geometry. First among these are definitions or agreements as to what certain terms such as parallel, perpendicular mean and we have dealt with them to some extent in § 2. Secondly, we have to assume the

existence of certain fundamental things such as points, lines and angles. Thirdly, we must take certain things (self-evident or otherwise) for granted, as they cannot be proved. These which have to be accepted for the moment on the authority of the teacher or intuitive experience are called **Postulates.**† A postulate literally means a demand—a demand made on the faith of the learner. The following postulates are important:

(i) *Magnitudes which can be made to coincide with one another are equal.*

(This is the basis of the method of '**Superposition**' *i.e.*, placing one geometrical entity such as a point, or a straight line on another in the process of comparing two geometrical figures).

(ii) *All right angles are equal.*

(This enables us to make the right angle a standard unit of measurement of angles. Attempts are sometimes made to prove this by the method of superposition which involves the further assumption of the rigidity of figures transferred from one place to another. The assumption of rigidity of figures is in effect equivalent to the assumption of equality under certain conditions).

(iii) *Two straight lines cannot enclose a space.*

(iv) *Two intersecting straight lines cannot both be parallel to a third straight line.*

(This is generally known as Playfair's axiom. It is not however Playfair's discovery, for it is distinctly stated by Proclus who lived more than 1300 years before Playfair.)

† Some of these postulates have already been referred to in § 2 as *facts*.

(v) *A straight line can be drawn from any one point to any other point.*

(vi) *A finite straight line can be produced to any length in the same straight line.*

(vii) *A circle can be drawn with any centre and radius.*

(The last three postulates merely imply the existence of straight lines of any length and circles of any radius. They do not imply anything as regards the actual drawing of figures or the instruments therefor. Nor do they declare that the only instruments we can use in geometry are the ruler for drawing and producing straight lines and the compasses for drawing circles, though these are convenient in practice. As a commentator of Euclid has put it, 'he would be a rash person, who taking things as they actually are, should postulate the drawing of straight lines from Aries to Libra'.)

(2) Methods and Exposition of Geometry.

The discussion of the subject-matter of Geometry proceeds by a chain of successive reasoning from definitions, axioms and postulates, and the results achieved at each stage are arranged into **Propositions.** **Propositions** are of two kinds, **Theorems** or **Problems**. A **Theorem** proves a geometrical truth and a **Problem** gives the construction of a geometrical figure.

The discussion of each proposition is logically arranged as follows:

(i) The **general enunciation** describing in general terms the geometrical truth to be investigated.

It consists of two parts, the **hypotheses or data** stating what is given, and the **conclusion** stating what is to be proved.

(ii) The **particular enunciation** which sets out in precise terms, with reference to a figure, the data and the conclusion.

(iii) The **construction** which explains the addition of more lines (circles and straight lines) to the figure which are necessary for the proof of the theorem or the construction of the geometrical figure required.

(iv) The **proof** which contains the arguments showing how the required object in the proposition has been accomplished.

If the hypothesis and the conclusion of a theorem be interchanged, the new theorem thus formed is said to be the **converse** of the first theorem.

A theorem with more than one hypothesis may have several converses obtained by interchanging one of the hypotheses with the conclusion.

The truth of a converse does not logically follow from the original theorem but must be proved separately. To take a familiar *example*: If all men are mortal, it does not necessarily follow that all mortals must be men.

A geometrical truth immediately following from a theorem is called a **corollary**. (According to Proclus, it is a result not directly sought but appearing as it were, by chance, without any additional labour and constituting as it is said, a sort of 'windfall' or 'bonus').

* There are certain standard methods of solution employed in proving geometrical truths or constructing geometrical figures. One method is the **reduction of one problem to another**, so that if the latter is solved, the former is also thereby solved. Thus the problem of

drawing a parallel to a given straight line, through a given point can be reduced to that of making an angle equal to a given angle. Another method is **mathematical analysis.** We quote the definition of Pappus, the Greek Geometer of the third century A.D.

'In **analysis**, we assume that which is sought as if it were already done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so re-tracing our steps we come upon something already known or belonging to the class of principles'. This is the method of discovery or the heuristic method. But the method of presentation of geometrical proofs generally follows just the opposite order and is known as '**Synthesis**'. To quote Pappus again, 'In synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of that which was sought.'

Then, there is the method of **reductio-ad-absurdum** which is a variety of analysis. Instead of starting with the required conclusion we start with its opposite and using the same process of analysis carry it back until we come upon something which is obviously false, or absurd, or contradictory to the hypothesis.

Besides the above, there are a few other methods of which a due account will be given in the following chapters as occasions arise.

CHAPTER II.

ANGLES AT A POINT.

§ 1. Theorem 1.

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

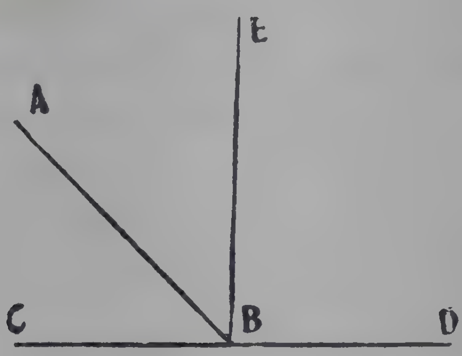


FIG. 41.

Particular Enunciation:

Let the straight line AB stand on the straight line CD at the point B forming the two angles ABC and ABD. It is required to prove that

$$A\hat{B}D + A\hat{B}C = 2 \text{ rt. } \angle\text{s.}$$

Construction: Let BE be drawn perpendicular to CD so that $C\hat{B}E = E\hat{B}D = 1 \text{ rt. } \angle$.

Proof: In the figure, $A\hat{B}D = A\hat{B}E + E\hat{B}D$ and $A\hat{B}C + A\hat{B}E = C\hat{B}E$.

$$\therefore \quad A\hat{B}D + A\hat{B}C + A\hat{B}E = A\hat{B}E + E\hat{B}D + C\hat{B}E.$$

From each of these equals take away ABE.

$$\therefore \quad A\hat{B}D + A\hat{B}C = E\hat{B}D + C\hat{B}E$$
$$= 2 \text{ rt. } \angle\text{s.}$$

Corollary 1 If a number of straight lines stand on the same straight line CD at the point A and on the same side of CD, the sum of the consecutive angles so formed is equal to two right angles; *i.e.*, in fig. (42), the sum of the angles 1, 2, 3, 4, 5 is equal to 2 rt. $\angle$s.

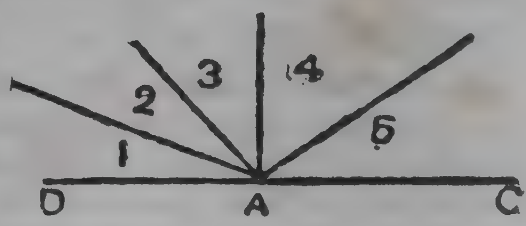


FIG. 42.

Corollary 2. A straight angle is equal to two right angles and therefore all straight angles are equal.

Corollary 3. If any number of straight lines meet at a point, the sum of all the angles made by consecutive lines is equal to four right angles; *i e*, in fig. (43), the sum of the angles = 4 rt. $\angle$s.

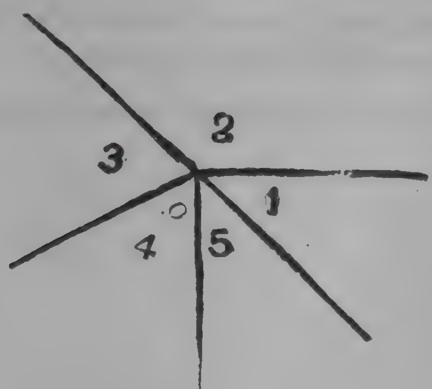


FIG 43.

Corollary 4. Two intersecting straight lines form four consecutive angles.

NOTE. In the above construction, we have introduced the straight line BE perpendicular to BD without showing how a perpendicular can be constructed. Such a construction is called a hypothetical construction, *i.e.*, a construction which is assumed to be possible. The use of hypothetical constructions enables us to group separately theorems and problems.

NOTE TO THE TEACHER. In connection with every proposition, it is advisable to make the pupils recapitulate the definitions of *terms* used, and the assumptions explicitly made, analyse the enunciation into two parts, hypothesis and conclusion, give alternative enunciations, draw important corollaries from the proposition, and finally explain the theoretical and practical applications. Thus in Theorem 1, the definitions may be required of the terms: straight line, angle, right angle; the enunciation may be analysed into

(*i*) *Hypothesis*: One straigut line standing on another.

(*ii*) *Conclusion:* Sum of the angles $= 2$ rt. $\angle$s.

The following alternative enunciations may be proposed :—

(1) If from a point in one straight line, another straight line be drawn on one side of it, the adjacent angles formed are supplementary.

(2) If the exterior arms of two adjacent angles are in one straight line, these angles are supplementary.

For practical and other applications, *vide* exercises given on the next page.

EXERCISE III.

1. There are four equal consecutive angles at a point. What is the magnitude of each?

2. If in the figure of Th. 1, $A\hat{B}D = 2\,A\hat{B}C$, what is the magnitude of each angle? Also show that $A\hat{B}C = 2\,A\hat{B}E$.

3. What is the angle which a minute division on a watch dial subtends at its centre?

4. Find the angle between the hour-hand and the minute-hand of a clock at (1) 6-15, (2) 8-30, (3) 10-45, (4) 5 o'clock.

5. Between 4-20 and 5 o'clock, the angle between the hands of a watch is observed to be 45°, what is the time?

6. The sides of a quadrilateral are produced in order, as in the figure and the exterior angles so formed are respectively equal to 60°, 120°, 110° and 70°. Find the magnitudes of the corresponding interior angles.

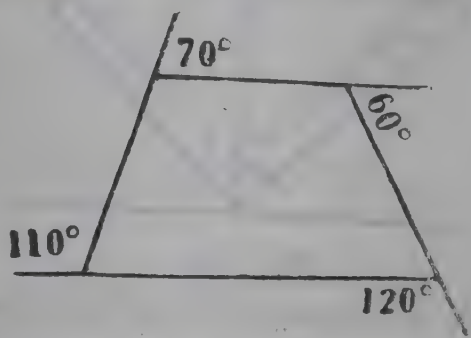


FIG. 44.

7. A straight line CD cuts two other straight lines as in the following figure, at G and H.

If $\hat{p} = \hat{p}' = 50°$, find the magnitudes of the angles q', r, and r'.

Also, show that $\hat{q} + \hat{p}' = \hat{r} + \hat{s}'$.

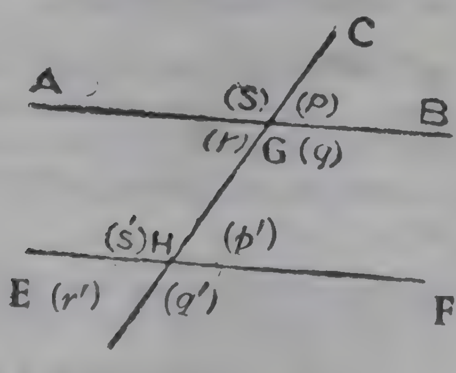


FIG. 45.

8. In the figure of Th. 1, find the angle between the bisectors of (i) the angles ABE, ABC,

(ii) „ ABC, ABD,

(iii) „ ABD, ABE.

9. *Def. The straight lines which bisect an angle and its adjacent supplementary angle are called the* **internal** *and* **external bisectors** *of the angle.*

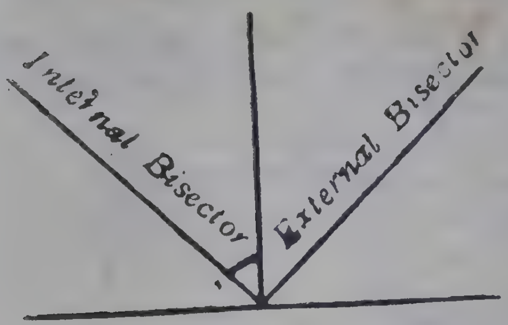


FIG. 46.

Thus in the figure of Th. 1, the bisectors of the angles ABC, ABD are the internal and external bisectors of the angle ABC.

Prove that the internal and external bisectors of an angle are at right angles to one another.

10. The corners at the ends of an edge of a rectangular sheet of paper are folded down so as to form two creases which meet at the edge and the two parts of the edge meet in the same straight line. Show that the creases are at right angles to one another.

11. Show how the proof given for Th. 1, has to be modified if BE lies within the angle ABC.

12. OA, OB, OC, OD are four straight lines, taken in (anti-clockwise) order. The angle BOC is a right angle; Prove that the angles AOB and COD are complementary or supplementary according as the angle AOD is a straight or a right angle. 10

13. OA, OB are two intersecting straight lines. OC, OD are drawn at right-angles to OA, OB respectively. Prove that $\hat{COD}$ is equal or supplementary to $\hat{AOB}$.

§ 2. Theorem 2. (Converse of Th. 1).

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.

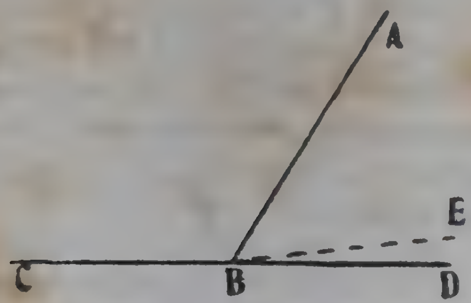


FIG. 47.

Particular Enunciation:

The sum of the two adjacent angles ABC, ABD is equal to two right angles.

It is required to show that the exterior arms BC, BD are in the same straight line.

Construction: Produce CB beyond B to E.

Proof: Since, by construction, CBE is a straight line,

$$A\hat{B}C + A\hat{B}E = 2 \text{ rt. } \angle\text{s.} \quad \text{(Th. 1.)}$$

But, by hypothesis,

$$A\hat{B}C + A\hat{B}D = 2 \text{ rt. } \angle\text{s.}$$

$$\therefore \quad A\hat{B}C + A\hat{B}E = A\hat{B}C + A\hat{B}D.$$

From each of these equals, take away $A\hat{B}C$.

$$\therefore \quad A\hat{B}E = A\hat{B}D.$$

Since these angles are on the same side of AB, BD and BE must coincide; *i.e.*, BD is in the same straight line as BC.

Corollary 1. Two different straight lines cannot have a common segment.

Corollary 2. At a given point in a straight line, only one straight line can be drawn perpendicular to it.

NOTE 1. *In Th. 2 it is given that* $A\hat{B}C + A\hat{B}D = 2$ *rt.* $\angle$*s and it is proved that BC, BD are in the same straight line: whereas in Th. 1, it is given that BC, BD are in the same straight line and it is proved that* $A\hat{B}C + A\hat{B}D = 2$ *rt.* $\angle$*s. Hence, Ths. 1 and 2 are converses of each other.*

NOTE 2. Very often in proving converse theorems, the original theorem is used. Thus, the proof of Th. 2 is based on Th. 1.

[A theorem follows from its converse, whenever the following principle known as the **Rule of Identity** applies: If there is but one *X* and but one *Y*, and we have the proposition that if a thing is *X*, it is also *Y*, the converse, *viz*, if a thing is *Y*, it is also *X*, necessarily follows from it. To take a familiar instance, if there be in a village only one Post Office and one Elementary School and it is known that the Post Office is the School, then it follows that the School is also the Post Office.]

NOTE 3. Th. 2 states a property of the straight line and not of the angles. It supplies a test by which we can examine whether three points are in the same straight line or whether a line is straight at any point.

EXERCISE IV.

1. What are the assumptions on which the proof of Th. 2 is based?

2. Give some alternative enunciations of Th. 2.

3. AOB, BOC, COD are three consecutive angles of magnitude 50°, 60°, 70°. Show that OA, OD are in the same straight line.

4. Show how you will use your protractor to test whether a line is straight.

5. Six lines meet at a point like the spokes of a wheel and make equal angles with one another all round the point. Prove that the six lines form three straight lines.

6. Enunciate and prove the converse of Ex. III, 9.

7. EHF is a straight line. HG is any straight line through H and angles HGB, HGA are made on opposite sides of HG equal respectively to the angles EHG and GHF. Show that AG and GB are in the same straight line.

8. *Def. When two straight lines AB, CD cross one another at O as in the figure, the pairs of angles on opposite sides of the common vertex are called* **vertically opposite angles.**

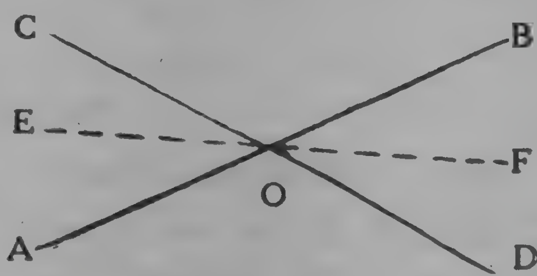


FIG. 48.

If $\angle BOC = 100^\circ$, what is the magnitude of each of the other three angles? Point out the equal pairs of angles.

If OE, OF be the bisectors of the vertically opposite angles AOC, BOD, prove that EO, OF are in the same straight line.

If E′F′ be drawn through O perpendicular to EF, prove that it bisects both the angles BOC and AOD.

9. What indirect measurements are made possible by Th. 1 and 2?

§ 3. Theorem 3.

If two straight lines intersect, the vertically opposite angles are equal.

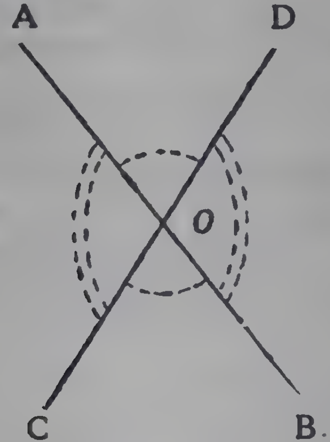


FIG. 49.

Particular Enunciation:

Let AB, CD be two straight lines intersecting at O so that the angles AOC and BOD are vertically opposite, as also the angles AOD and BOC. It is required to prove that

$$A\hat{O}C = B\hat{O}D \text{ and } A\hat{O}D = B\hat{O}C.$$

Proof: AO stands on the straight line CD;

$\therefore \quad A\hat{O}C + A\hat{O}D = 2\text{rt. } \angle\text{s.} \quad$ (Th. 1.)

Again DO stands on the straight line AB;

$\therefore \quad A\hat{O}D + B\hat{O}D = 2\text{rt. } \angle\text{s.} \quad$ (Th. 1.)

Hence $A\hat{O}C + A\hat{O}D = A\hat{O}D + B\hat{O}D$.

From these equals, take away $A\hat{O}D$.

$\therefore \quad A\hat{O}C = B\hat{O}D$.

Similarly, it can be shown that $A\hat{O}D = B\hat{O}C$.

EXERCISE V.

1. Prove the above theorem taking BO to stand on CD and CO on AB.

2. If in the figure of Th. 3, $\angle AOC = x^\circ$, express the magnitudes of the other angles in terms of x.

3. EF is a straight line cutting AB, CD at G and H respectively so that $\angle EGB = \angle GHD$. Prove that $\angle AGH = \angle GHD$, $\angle BGH = \angle FHD$, and $\angle FHC + \angle AGE = 2$ rt. $\angle$s.

4. Enunciate and prove the converse of Th. 3.

5. *Prove that the bisectors of vertically opposite angles are in the same straight line.*

6. Prove Th. 3 practically, by taking two rods hinged together in the middle. Give some familiar illustrations of such hinged rods.

7. OA, OB, OC, OD are four straight lines taken in order round a point such that $\angle AOB = \angle COD$ and $\angle BOC = \angle AOD$. Prove that AO, OC form one straight line and BO, OD another straight line.

8. *A straight line OC stands on the straight line AB at O. Prove that every other straight line through O (produced, if necessary) must lie within the angle AOC or COB.*

HISTORICAL NOTE. This proposition about the vertically opposite angles is nearly 2500 years old. It seems to be the first proposition needing a proof, while the two preceding ones are so evident as to be taken for granted. From Proclus (410 – 485 A.D.) the great commentator of Euclid, we learn that Eudemus who wrote a valuable History of Geometry (now unfortunately lost) attributed the discovery of this theorem to Thales of Miletus (640 – 548 B.C.) one of the seven Wise Men of Greece. But Euclid was the first to give a scientific proof of the theorem, though Thales was the first to introduce the principles of geometry into Greece and discovered much himself.

CHAPTER III.

PARALLEL STRAIGHT LINES.

§ 1. Preliminary Notions.

Def. A straight line which cuts two or more straight lines is called a ***Transversal.***

When a transversal cuts two straight lines, eight angles are formed, which can be grouped into pairs having special names.

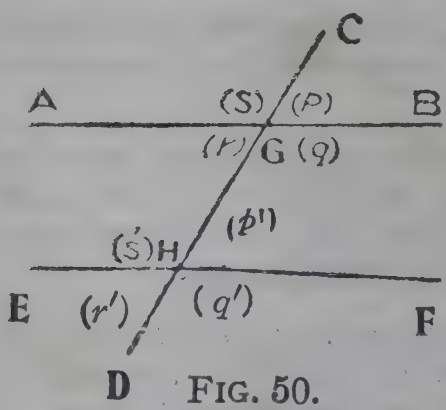


FIG. 50.

In fig. 50 the pairs of angles (p, p'), (q, q'), (r, r'), (s, s') are called **corresponding** angles; the pairs (p', r) and (q, s') are called **alternate** angles; the angles (q, p') are said to be **interior** angles on the same side of the transversal; so also (r, s').

Playfair's Axiom: **Two intersecting straight lines cannot both be parallel to a third straight line.**

It is instructive to set down here the Parallel-Postulate as enunciated by Euclid and point out the relation between, and the relative merits of, the two postulates:

If a straight line falling on two other straight lines make the interior angles on the same side together less than two right angles, the two straight lines will, if produced infinitely, meet on that side on which the angles are together less than two right angles.

As Dodgson (*Euclid and his Modern Rivals*, pp. 44-46) remarks, Euclid's postulate puts before the beginner clear and

positive conceptions, a pair of straight lines, a transversal, and two angles together less than two right angles, whereas Playfair's axiom requires him to realise the impossibility of a pair of intersecting straight lines which never meet a third straight line, however far produced—a negative conception which does not convey to the mind any clear notion of the relative positions of the lines.

Further Playfair's axiom involves really more than Euclid's postulate, so that, to that extent, it is an unnecessary strain on the faith of the learner. To deduce Playfair's axiom from Euclid's postulate, we require in addition to the postulate a number of theorems.

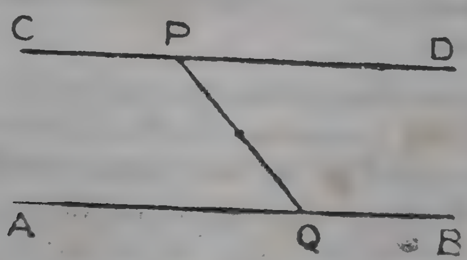


FIG. 51.

From Euclid's postulate it is easy to infer as shown below that through a given point there cannot be drawn† *more than one* straight line parallel to a given straight line.

Let P be a given point and AB a given st. line. Take any point Q in AB. Join PQ. Through P, obviously one straight line CPD can be drawn so that $\angle CPQ = \angle PQB$.‡

Now $\angle PQB + \angle QPD = \angle CPQ + \angle QPD = 2$ rt. $\angle$s
and $\angle CPQ + \angle PQA = \angle PQB + \angle PQA = 2$ rt. $\angle$s.

Any other st. line (say C′D′) through P must lie either within the angle CPQ or QPD and in either case, two interior angles on

† This implies that there may be either *one* straight line or *none* parallel to a given straight line.

‡ In Playfair's axiom, CD is parallel to AB and every straight line which cuts CD cuts AB also.

the same side of PQ made by AB, C'D' are together less than two right angles and therefore every straight line through P other than CD must meet AB. Hence if CD does not meet AB, it is the only straight line which can be parallel to AB. (In Th. 4, it will be proved that CD does not meet AB.)

The only merit in Playfair's axiom is its brevity and apparent simplicity.

Of late there has been an attempt to substitute for Playfair's axiom, another postulate known as the **Postulate of Similarity** *viz.*

'**Any figure can be reproduced anywhere on any enlarged or diminished scale**', and to deduce from it the theory of similar figures and parallels.

This postulate of similarity has the sanction of eminent mathematicians like Wallis, Laplace and Clifford as well as great educational and mathematical thinkers of modern times like Prof. T. P. Nunn and M. J. M. Hill. Its claims from the teaching standpoint are also quite undoubted. 'For no school-boy could be made to doubt (for instance) that a perfect model of his favourite locomotive engine or battleship could be made on any scale whatever. To him it is a truth quite as evident as Playfair's axiom and much more interesting.' If the principle of similarity should not be true, all our faith in scale-drawing and with it our surveying and engineering calculations must go.

But the difficulty in adopting this postulate is that it involves a thorough revolution, nay, inversion, in the sequence hallowed by centuries of acceptance,—a revolution which may not be quite agreeable to many conservative teachers and educational authorities who are generally sceptic about things novel

§ 2. Theorem 4.

(*1*) *When a straight line cuts two other straight lines so as to make a pair of alternate angles equal, then the two straight lines are parallel.*

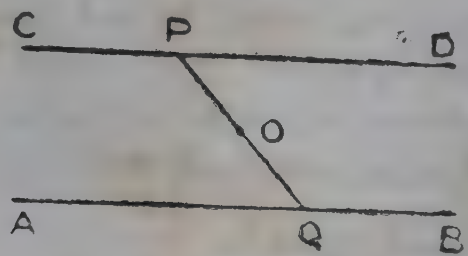


FIG. 52.

Particular Enunciation:

Let CD, AB be two straight lines and PQ cut them so that $\angle CPQ = \angle PQB$ (alt. angles).

It is required to prove that CD and AB are parallel.

Proof: Since $\angle CPQ = \angle PQB$, their supplements are also equal, *viz.*, $\angle DPQ = \angle PQA$.

Let O be the mid. pt. of PQ. Turn the figure CPQA about O, in the same plane so that P and Q interchange places, which is possible since OP = OQ.

Since $\angle CPQ = \angle PQB$, PC falls on QB

and since $\angle PQA = \angle DPQ$, QA falls on PD,

so that if PD and QB meet when produced towards D and B, QA and PC which are made to coincide with them must also meet.

Now it may be assumed that QA and PC carry with them this property when they go back to their original positions.

Thus if CD, AB meet in one direction (say, on the side of D, B), they must also meet in the other direction, *i e.*, two straight lines will enclose a space, which is impossible.

$\therefore$ CD and AB are parallel.

(2) *When a straight line cuts two other straight lines, so as to make* (i) *a pair of corresponding angles equal or* (ii) *a pair of interior angles on the same side together equal to two right angles, then the two straight lines are parallel.*

Particular Enunciation :

Let CD, AB be two straight lines cut by EPQF at P, Q so that (i) $\angle$EPD = $\angle$EQB

or (ii) $\angle$DPQ + $\angle$PQB = 2 rt. $\angle$s.

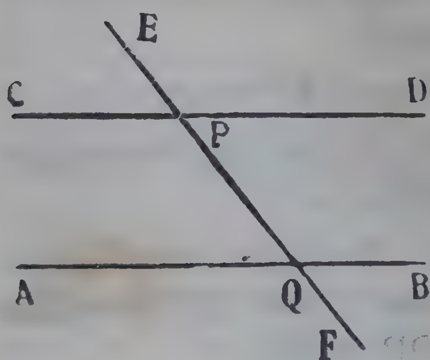


FIG. 53.

It is required to prove that CD and AB are parallel.

Proof: (i) $\angle$EPD = $\angle$CPQ (vert. opp. $\angle$s.)

But $\angle$EPD = $\angle$EQB (Hypothesis).

$\therefore$ $\angle$EQB = $\angle$CPQ.

These are alt. $\angle$s.

$\therefore$ CD and AB are parallel by (1) above.

(ii) $\angle$DPQ + $\angle$CPQ = 2 rt. $\angle$s (adj. $\angle$s made by QP with CD).

But $\angle$DPQ + $\angle$PQB = 2 rt. $\angle$s. (Hypothesis).

$\therefore$ $\angle$DPQ + $\angle$CPQ = $\angle$DPQ + $\angle$PQB.

$\therefore$ $\angle$CPQ = $\angle$PQB. These are alt. $\angle$s.

$\therefore$ CD and AB are parallel by (1) above.

Cor. Two straight lines perpendicular to a third straight line at different points on it are parallel to one another.

EXAMPLE. *If one side of a triangle be produced, the exterior angle so formed is greater than either interior opposite angle.*

Let ABC be a triangle of which the side BC is produced to D. It is required to prove that ∠ACD is greater than either of the interior opposite angles BAC, ABC.

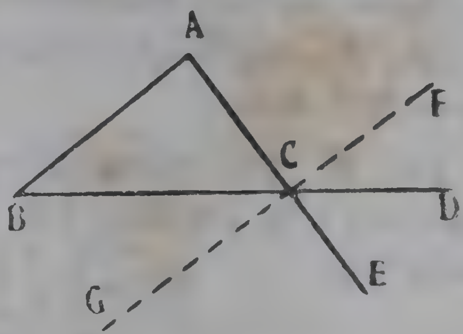


FIG. 54.

Construction: Produce AC to E. At C, on the side of AC opposite to B make ∠ACF = ∠BAC; and at C in BC, on the side of BC opposite to A, make ∠BCG = ∠ABC.

Proof: Since ∠ACF = ∠BAC (alt. ∠s).

∴ CF is parallel to BA (Th. 4).

∴ CF cannot lie within the angle DCE; for, then CF will fall when produced within ∠BCA and cut AB.

Hence it must lie within the angle ACD.

∴ ∠ACD is greater than ∠ACF, *i.e.*, ∠BAC.

Similarly CG must lie within the angle BCE.

∴ ∠BCE is greater than ∠BCG, *i.e.*, ∠ABC.

But ∠BCE = ∠ACD (vert. opp. ∠s).

∴ ∠ACD is greater than either of the int. opp. ∠s BAC, ABC.

Cor. 1. Since $\angle ACD$ is greater than $\angle ABC$,

$\therefore$ $\angle ACD + \angle ACB$ is greater than $\angle ABC + \angle ACB$.

$\therefore$ $\angle ABC + \angle ACB$ is less than 2 rt. $\angle$s.

Similarly, it can be shown that $\angle BAC + \angle ACB$ is less than 2 rt. $\angle$s.

By using the exterior angle at A, it can be shown that $\angle ABC + \angle BAC$ is also less than 2 rt. $\angle$s.

Hence, we have the theorem :

Any two angles of a triangle are together less than two right angles.

Cor. 2. The three angles of a triangle are together less than 3 rt. $\angle$s. For, if A, B, C be the three angles, we know that

$$A + B < 2 \text{ rt. } \angle\text{s},$$

$$B + C < 2 \text{ rt. } \angle\text{s},$$

and

$$C + A < 2 \text{ rt. } \angle\text{s}.$$

$\therefore$

$$2(A + B + C) < 6 \text{ rt. } \angle\text{s},$$

i.e.,

$$A + B + C < 3 \text{ rt. } \angle\text{s}.$$

EXERCISE VI.

1. Prove that the straight lines AB, CD in the fig. of Th. 4 will be parallel if either

   (i) $\angle CPE = \angle BQF$ or

   (ii) $\angle AQF = \angle EPD$ or

   (iii) $\angle CPE + \angle AQF = 2$ rt. $\angle$s.

2. Take a ruler. Place a set-square with its longest edge along the square edge of the ruler and holding the ruler firmly, slide the set-square along it into various positions. If you rule along another edge of the set-square in these various positions, show that you get parallel straight lines.

3. Use your set-square and ruler to draw a straight line through a given point P parallel to a given straight line AB.

4. With the help of Euclid's parallel postulate and Th. 4, prove Playfair's axiom.

5. Conversely, from Playfair's axiom and Th. 4, deduce Euclid's parallel postulate.

6. If in a quadrilateral, two pairs of adjacent angles are supplementary, show that the opposite sides are parallel.

7. In the figure of Th. 4, show that the bisectors of the angles at P and Q form a rectangle.

8. Prove that no triangle can have either two obtuse angles or two right angles or one obtuse angle and one right angle. Hence, deduce that in every triangle, there are at least two acute angles.

9. Prove that a pentagon cannot have all its angles right.

(*Proof*: If possible, let a, b, c, d, e be the consecutive sides of a pentagon having all its angles right. Then $a \perp b$, $b \perp c$, $c \perp d$, $d \perp e$. $\therefore a \parallel e$, *i.e.* a and e cannot meet and the pentagon cannot be closed).

10. Can we infer from Playfair's axiom alone that through a given point *one and only one* straight line can be drawn parallel to a given straight line?

§ 3. Theorem 5. (Converse of Th. 4).

If a straight line cuts two parallel straight lines, it makes (1) alternate angles equal, (2) corresponding angles equal, and (3) the interior angles on the same side of the line together equal to two right angles.

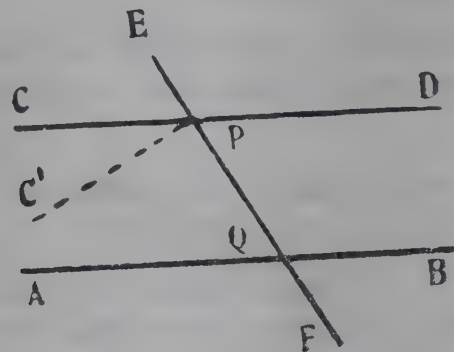


FIG. 55.

Particular Enunciation:

Let CD, AB be two parallel straight lines cut by **EF** at P, Q.

It is required to prove

(i) $\angle CPQ = \angle PQB$

(ii) $\angle EPD = \angle PQB$

and (iii) $\angle DPQ + \angle PQB = 2$ rt. $\angle$s.

Construction:

If $\angle CPQ$ be not equal to $\angle PQB$, draw PC′ so that $\angle C'PQ$ is equal and alternate to $\angle PQB$.

Proof:

By construction, $\angle C'PQ = \angle PQB$ (alt. $\angle$s).

$\therefore$ C′P is parallel to AB.

By hypothesis, CP is parallel to AB.

$\therefore$ Two intersecting straight lines CP, C$'$P are both parallel to AB, which is contradictory to Playfair's axiom

$\therefore$ $\angle$CPQ cannot be unequal to $\angle$PQB

i.e. $\angle CPQ = \angle PQB$ ... ... (i)

But $\angle CPQ = \angle EPD$ (vert. opp. $\angle$s).

$\therefore$ $\angle EPD = \angle PQB$ ... ... (ii)

Again, $\angle CPQ + \angle DPQ = 2$ rt. $\angle$s (adj. $\angle$s made by QP with CD) and $\angle CPQ = \angle PQB$ (proved.)

$\therefore$ $\angle DPQ + \angle PQB = 2$ rt. $\angle$s ... (iii)

Cor. If a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

EXAMPLE : *If straight lines are drawn from a point parallel to the arms of a given angle, the angle between these straight lines is equal or supplementary to the given angle.*

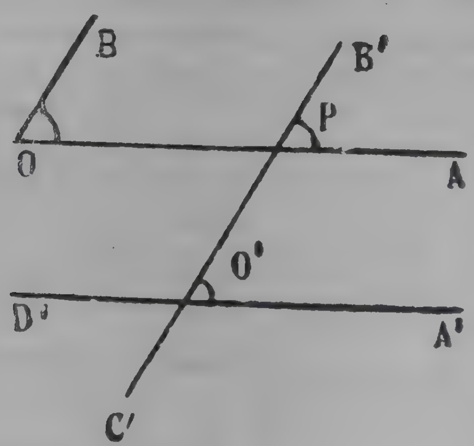


FIG. 56.

Let AOB be the given angle and A$'$D$'$, B$'$C$'$ two other lines drawn through a pt. O$'$ parallel to OA, OB respectively. It is required to prove $\angle A'O'B' = \angle C'O'D' = \angle AOB$ and $\angle A'O'C' = \angle B'O'D' =$ the supplement of $\angle$AOB.

Construction : Let O$'$B$'$ cut OA at P.

Proof: Since OB, O′B′ are parallel and OA cuts them,

$\therefore\quad \angle APB' = \angle AOB$ (corresponding angles).

Again, since OA, O′A′ are parallel and O′B′ cuts them,

$\therefore\quad \angle APB' = \angle A'O'B'$ (corresponding angles).

$\therefore\quad \angle A'O'B' = \angle AOB.$

But $\angle C'O'D' = \angle A'O'B'$ (vert. opp. $\angle$s)

and $\angle A'O'C' = \angle B'O'D'$ = the supplement of $\angle A'O'B'$.

$\therefore\quad \angle A'O'B' = \angle C'O'D' = \angle AOB$ and

$\angle A'O'C' = \angle B'O'D'$ = the supplement of $\angle AOB$.

NOTE 1. From a comparison of the data and conclusions of Th. 5 with those of Th. 4, it is seen that they are converses of each other.

NOTE 2. The proof of Th. 5 depends upon Th. 4 and Playfair's axiom. As remarked already in Note 2, p. 54, a theorem follows from its converse whenever the **Rule of Identity** applies, and this identity is here supplied by Playfair's axiom.

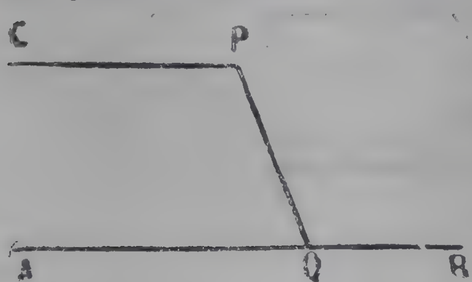


FIG. 57.

If AB be a given straight line, P a given point outside it, and PQ any straight line from P to AB meeting it at Q, only *one* straight line CP can be drawn thro' P so that $\angle CPQ$ is equal and alternate to $\angle PQB$ and by Playfair's axiom, only *one* straight line goes thro' P parallel to AB. Now Th. 4 identifies the line CP with the parallel to AB. Hence by the **Rule of Identity**, the parallel CP must make $\angle CPQ = \angle PQB$.

HISTORICAL NOTE. It was for a long time thought that the parallel postulate of Euclid or any of its equivalent forms was capable of proof. Most curiously enough, the attempts to prove the postulate led incidentally to develop a consistent sequence of

theorems based on a denial of the above postulate and laid the foundations of what are called non-Euclidean geometries. The logical possibility of these geometries showed for the first time that the postulate of parallels could never be proved. In 1868, Prof. Eugenio Beltrami, an Italian mathematician, proved the actual indemonstrability of the postulate. This fact which posterity took so much time to settle, the genius of Euclid must have perceived by a stroke of remarkable intuition, and it is admirable how he concluded that his epoch-making postulate, which was necessary for the validity of his geometrical structure, must ever remain a postulate, *i.e.*, incapable of proof.

EXERCISE VII.

1. Deduce Th. 5 from Th. 4 and Euclid's parallel postulate.

2. One angle of a parallelogram is 50°. Find the magnitudes of the other angles.

Prove that the adjacent angles of a parallelogram are supplementary and the opposite angles are equal.

3. PQRS is a quadrilateral in which PQ is parallel to RS. Prove that the interior angles of the quadrilateral are together equal to four right angles.

4. ABC is a triangle in which $B = 50°$, $C = 60°$. Through A, a straight line PAQ is drawn parallel to BC. Find, without measurement, the magnitudes of the angles PAB, QAC and hence deduce the magnitude of the angle BAC.

5. ABCDEF is a hexagon in which the opposite sides (AB, DE), (BC, EF), and (CD, FA) are parallel. Prove that the opposite angles are equal, *viz.*,

$$\angle A = \angle D, \angle B = \angle E, \text{ and } \angle C = \angle F.$$

6. **If straight lines are drawn from a point perpendicular to the arms of an angle, the angle between these straight lines is equal or supplementary to the given angle.**

7. If two triangles have two angles of the one equal to two angles of the other each to each, show that the third angles are equal. (Hint: Apply one triangle to the other).

8. ABC, DEF are two equal angles. If AB is parallel to DE, is BC parallel to EF?

9. P is a point at a distance of 1″ from a given straight line AB. Thro' P draw two straight lines PQ, PR to meet AB at Q, R respectively such that $\angle PQR = \angle PRQ = 70^\circ$.

10. Prove that straight lines, which are parallel to the same straight line, are parallel to one another.

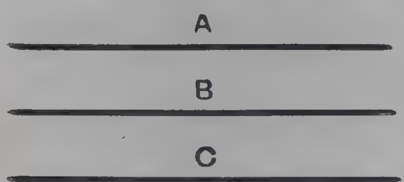


FIG. 58.

(*Proof*: Let A and B be two st. lines which are parallel to a third st. line C. Then A and B cannot intersect since it will contradict Playfair's axiom; hence A is parallel to B.)

NOTE 1. In Euclidean Geometry, two straight lines in the same plane either intersect or are parallel. There is no third alternative of non-intersecting, non-parallel straight lines.

NOTE 2. Each of two parallel straight lines lies wholly on one side of the other straight line.

NOTE 3. If a straight line cuts one of a system of parallel straight lines, it cuts the rest also.

11. Assuming the first part of Th. 5, as an axiom, prove Playfair's axiom and Euclid's postulate of parallels.

CHAPTER IV.

ANGLES OF TRIANGLES AND POLYGONS.

§. 1. Theorem 6.

The three angles of a triangle are together equal to two right angles.

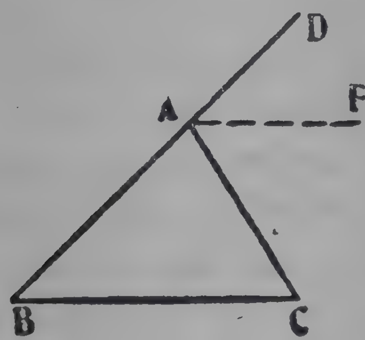


FIG. 59.

Let ABC be a triangle.

It is required to prove that

$$\angle ABC + \angle BCA + \angle BAC = 2 \text{ rt. } \angle s.$$

Construction: Thro' A draw AF parallel to BC and produce BA to D.

Proof: Because AF and BC are parallel and DB, AC meet them,

$\therefore$ $\angle DAF = \angle ABC$ (corresp. $\angle$s)

and $\angle FAC = \angle BCA$ (alt. $\angle$s).

$\therefore$ $\angle DAF + \angle FAC = \angle ABC + \angle BCA$

i.e. $\angle DAC = \angle ABC + \angle BCA.$

To each of these equals, add $\angle BAC$

Then $\angle DAC + \angle BAC = \angle AEC + \angle BCA + \angle BAC.$

But, since BAD is a straight line, the angles at A are together equal to two right angles.

$\therefore$ $\angle ABC + \angle BCA + \angle BAC = 2$ rt. $\angle$s,

i.e. the sum of the angles of the triangle is equal to two right angles.

Cor. 1. If one side of a triangle be produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

Thus, we proved that $\angle DAC = \angle ABC + \angle BCA$.

It follows from the above that an exterior angle is greater than either of the interior opposite angles.

It leads to the further corollary that any two angles of a triangle are together less than two right angles.

N.B. It may be noted that the last two results were proved before independently of Playfair's axiom and it is obvious that they do not imply Th 6.

Cor. 2. (i) No triangle can have two angles right or obtuse, or one angle right and the other obtuse. But every triangle must have at least two acute angles.

(ii) If a triangle has not all its angles equal, at least one angle must be less than 60° and another greater than 60°.

(iii) Only one perpendicular can be drawn to a straight line from an external point.

Cor. 3. If two triangles have two angles of the one equal to two angles of the other each to each, then the third angles are equal.

Cor. 4. If the sides of a triangle be produced in order, the sum of the exterior angles so formed is equal to twice the sum of the interior angles, *i.e.* 4 rt. $\angle$s.

NOTE. This theorem enunciates an important condition for the existence of a triangle with three given angles. No triangle is possible whose angles are not together equal to two right angles. It is well to notice that Euclid's parallel postulate is a converse of the theorem that any two angles of a triangle are together less than two rt. $\angle$s.

EXERCISE VIII.

1. Cut out the corners of a card-board triangle and fit them together at one point. What do you notice?

2. Repeat the process of Ex. 1 for a card-board quadrilateral and prove your conclusion theoretically.

3. Which is the greatest angle in a right-angled triangle? Show that if the sum of two angles of a triangle is equal to the third angle, the triangle must be right-angled.

In which triangle is the sum of any two angles greater than the third?

4. Can the sum of any two angles of a triangle be equal to twice the third angle?

5. If two triangles have the angles of the one equal or supplementary to the angles of the other, each to each, prove that the triangles must be equiangular.

6. If two triangles have three sides of the one parallel to the three sides of the other, each to each, show that the triangles are equiangular.

7. If two triangles have three sides of the one perpendicular to the three sides of the other, each to each, show that the triangles are equiangular.

8. Prove that any three angles of a convex polygon are together greater than two right angles.

9. The internal bisectors of the angles B and C of a triangle ABC meet at I and the external bisectors at I_1. Prove

that $\angle BIC = 90° + \frac{A}{2}$ and $\angle BI_1C = 90° - \frac{A}{2}$. Hence deduce that BC subtends supplementary angles at I and I_1.

10. P is any point within the triangle ABC. Prove that $\angle BPC = \angle BAC + \angle ABP + \angle ACP$.

11. By drawing the diagonals through a vertex of a pentagon, prove that the sum of the interior angles is equal to six right angles.

12. **If the opposite angles of a quadrilateral be equal, prove that it is a parallelogram.**

13. **If in a quadrilateral one pair of opposite angles be supplementary, prove that the other pair also must be supplementary. If one side of such a quadrilateral is produced, prove that the exterior angle so formed is equal to the interior opposite angle.**

14. The bisectors of the angles B and C of a triangle ABC meet the opposite sides at D and E respectively.

(i) If $\angle ADB$ and $\angle AEC$ are supplementary, prove that $\angle BAC = 60°$.

(ii) If $3 \angle ADB = 2 \angle AEC = 120°$, find A, B, and C.

(iii) Show that $\angle ADB$ can never be equal to twice $\angle AEC$.

15. By joining the diagonals of a quadrilateral, prove that the sum of the exterior angles of a quadrilateral (taken one at each vertex) is equal to the sum of the interior angles.

16. **If in a quadrilateral, one pair of opposite angles be equal, prove that the bisectors of the other pair of opposite angles are either parallel or coincident.**

§ 2. Extension to Polygons.

Theorem 7.

If the sides of a convex polygon be produced in order, the sum of the exterior angles so formed is equal to four right angles.

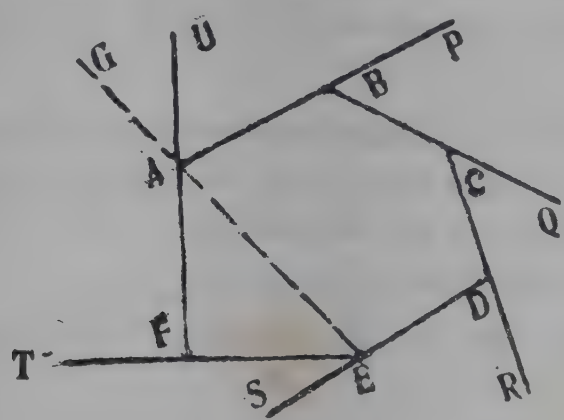


FIG. 60.

Let ABCDEF be a convex polygon; its sides AB, BC, CD, DE, EF, FA are produced beyond the vertices B, C, D, E, F, A in order to P, Q, R, S, T, U respectively, to form exterior angles.

It is required to show that the sum of these angles = 4 rt. $\angle$s.

Construction: Join EA and produce it to G.

Proof:

$$\angle GAB = \angle GAU + \angle UAB$$
$$= \angle FAE + \angle UAB$$

and

$$\angle SEA = \angle SEF + \angle FEA.$$

$$\therefore \angle SEA + \angle GAB = \angle SEF + \angle FEA + \angle FAE$$
$$+ \angle UAB = \angle SEF + \angle TFA + \angle UAB,$$

since $\angle$TFA is an ext. $\angle$ of the triangle AFE.

Add to these equals the sum of the ext. $\angle$s at B, C, D.

Then the sum of the ext. ∠s of the polygon ABCDE = the sum of the ext. ∠s of the polygon ABCDEF.

Thus, it is seen that the sum of the ext. ∠s of any polygon of n sides is the same as the sum of ext. ∠s of a polygon of $(n-1)$ sides. By repeated application of this result, we find that the sum of the ext. ∠s of the polygon is finally equal to the sum of the ext. ∠s of the triangle ABC.

Since each ext. ∠ of a triangle is the supplement of the corresp. int. ∠, and the sum of the three int. ∠s is equal to 2 rt. ∠s, the sum of the exterior angles must be equal to 4 rt. ∠s (*vide* Th. C, Cor. 4.)

Hence the sum of the ext. ∠s of the polygon is also equal to 4 rt. ∠s.

N.B. 1. This property is true only of a convex polygon, *i.e*, one in which the join of every pair of non-adjacent vertices lies entirely within the polygon. In such a polygon, every interior angle is less than two right angles.

2. Since the two ext. ∠s at any vertex are equal, the proposition will be true, even if the sides are *not* produced *in order*, but produced in such a way that only *one* ext. ∠ is formed at each vertex.

Cor. The sum of the interior angles of any convex polygon of n sides is equal to $(2n - 4)$ right angles.

For, the sum of an ext. ∠ and the int. ∠ at each vertex = 2 rt. ∠s.

∴ The sum of the int. ∠s of the polygon together with the sum of the corresponding ext. ∠s = $2n$ rt. ∠s.

But the sum of the ext. ∠s = 4 rt. ∠s.

∴ The sum of the int. angles = $(2n-4)$ rt. ∠s.

NOTE. The int. ∠s of regular rectilineal figures of 3, 4, 5, 6, 7, 8, 9, 10 sides are respectively 60°, 90°, 108°, 120°, $128\frac{4}{7}$°, 135°, 140°, 144°.

Example 1. An interior angle of a regular polygon is 144°. Find the number of sides in the polygon.

Let n be the number of sides.

Since all the int. ∠s are equal and their sum is $(2n - 4)$ rt. ∠s, the magnitude of each angle is $\left(\frac{2n-4}{n}\right)$ rt. ∠s.

$$\therefore \quad \frac{2n-4}{n} = \frac{144}{90}.$$

$$\therefore \quad 180n - 360 = 144n.$$

$$\therefore \quad 36n = 360,$$

i.e. $n = 10$.

NOTE. In order that n may be an integer, the supplement of the int. ∠ must be a sub-multiple of 360°.

Example 2. Show that it is impossible to fill the space about a point with more than six regular polygons.

As before, we find that an int. ∠ of a regular polygon is

$$\left(\frac{2n-4}{n}\right) \text{ or } \left(2 - \frac{4}{n}\right) \text{rt. ∠s.}$$

Now, $\left(2 - \frac{4}{n}\right)$ increases as n increases, *i.e.*, the magnitude of an interior angle of a regular polygon increases with the number of sides.

Thus, we get the least angle, when $n = 3$, and the least angle is the angle of an equil. Δ, *viz.* 60°. Since the sum of the angles at a point is equal to 360°, *six* equilateral triangles placed consecutively will just fill the space about a point and no more.

And, if we put in more regular polygons at a point, the sum of the angles will exceed 360°.

Thus, only six equilateral triangles and no other six regular figures can fill the space about a point.

Example 3. If five regular rectilineal figures can fill the space about a point, prove that they must be either four equilateral triangles and a hexagon, or three equilateral triangles and two squares.

Let A, B, C, D, E denote the five interior angles of five different regular figures placed consecutively about a point to fill the space about it.

Then $\quad A + B + C + D + E = 360°.$

If all these angles are equal, each $= 72°$. But 72° cannot be an int. angle of any regular polygon.

∴ Some of the angles must be unequal and in this case, let the least angle be A.

Then $\quad A < \frac{1}{5} \times 360°,$

i.e. $\quad < 72°.$

Now, the only int. ∠ of a regular figure less than 72° is 60°.

$$\therefore \quad A = 60°.$$

$$\therefore \quad B + C + D + E = 300°$$

As before, if all these angles are equal, each $= 75°$, which again does not correspond to any regular polygon.

∴ Some of the angles must be unequal and in this case, let the least angle be B.

Then $\quad B < \frac{1}{4} \times 300°,$

i.e. $\quad < 75°.$

The only int. ∠ of a regular figure less than 75° is 60°.

$$\therefore \quad B = 60°.$$

Hence $C + D + E = 240°$.

Again, if $C = D = E$, each $= 80°$ which is not an int. $\angle$ of any regular figure.

$\therefore$ The angles cannot all be equal.

Let the least angle be C.

Then $C < \frac{1}{3} \times 240°$; *i.e.* $< 80°$.

As before, $C = 60°$. $\therefore$ $D + E = 180°$.

If $D = E$, each $= 90°$.

If $D \neq E$, let the lesser angle be D.

Then $D < \frac{1}{2} \times 180°$, *i.e.* $< 90°$.

$\therefore$ $D = 60°$ and $E = 120°$.

Thus, either $A = B = C = 60°$, $D = E = 90°$ (i)
which correspond to 3 equil. Δs and 2 squares

or $A = B = C = D = 60°$ and $E = 120°$ (ii)
which correspond to 4 equil. Δs and 1 hexagon.

The following figures illustrate some patterns corresponding to (i) and (ii) above:

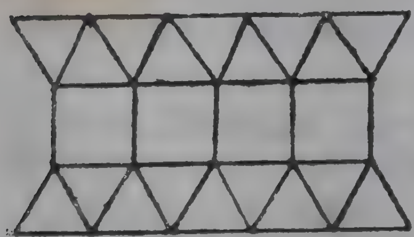


FIG. 61.

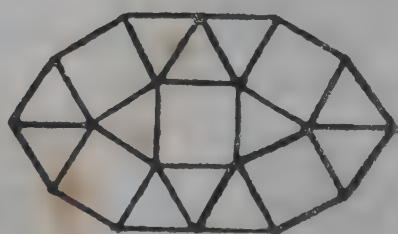


FIG. 62.

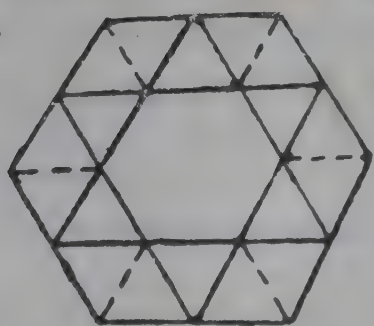


FIG. 63.

EXERCISE IX.

1. Draw any pentagon. Take a point within it and join the point to the vertices of the pentagon.

How many triangles are formed? What is the sum of all the angles of these triangles? What is the sum of all the angles at the interior point?

Frame the equation connecting the sum of the angles of the triangles and the sum of the interior angles of the pentagon. Hence deduce that the sum of the interior angles of the pentagon is equal to 6 rt. $\angle$s.

Generalise the above method for finding the sum of the int. $\angle$s of any polygon.

2. Take any hexagon. Join any vertex of the hexagon to the other vertices except the two adjacent to the vertex. How many triangles are formed? State the relation between the sum of all the angles of these triangles and the sum of the interior angles of the hexagon. Hence deduce that the sum of the interior angles of the hexagon is equal to 8 rt. $\angle$s.

Generalise the above method for finding the sum of the int. $\angle$s of any polygon.

3. Assuming that the sum of the interior angles of a polygon of n sides is equal to $(2n - 4)$ rt. $\angle$s, prove that the sum of the int. angles of a polygon of $(n + 1)$ sides is equal to $2(n + 1) - 4$ rt. $\angle$s and hence deduce that the formula is true for every integral value of n, not less than 3.

4. Assuming the sum of the interior angles of a polygon of n sides to be $(2n - 4)$ rt. $\angle$s, prove that the sum of the ext. $\angle$s, one at each vertex, is equal to 4 rt. $\angle$s.

5. Find the number of degrees in an interior angle of a regular polygon of

(i) 15 sides; (ii) 40 sides; (iii) 90 sides;
(iv) n sides; (v) $2n$ sides; (vi) $4n$ sides.

6. An angle of a regular polygon is (i) $1\frac{1}{5}$ rt. $\angle$; (ii) 165°; (iii) 1 rt. $\angle$, 75°, $36'$.

Find the corresponding number of sides.

7. Show that the interior angles of a pentagon cannot be in the ratio $1:2:3:4:5$.

8. Find the sum of the interior angles of the following figures:

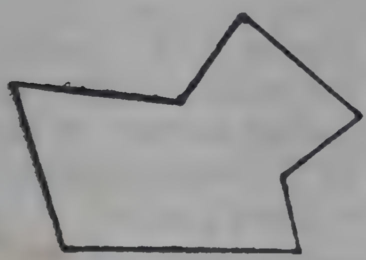


FIG. 64.

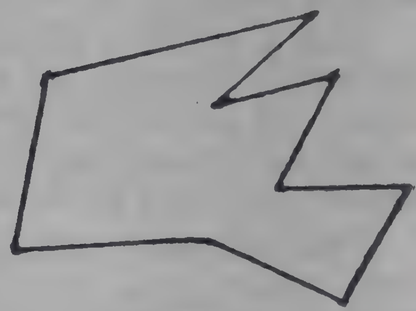


FIG. 65.

9. An interior angle of a regular polygon of n sides is *six* times an exterior angle of a regular polygon of $3n$ sides. Find n.

10. An interior angle of a regular polygon is supplementary to an interior angle of another regular polygon. Find the number of sides in each polygon.

11. Show that a tessellated pavement can be formed of tiles in the shape of regular polygons arranged round a point in any of the following ways:—(i) 6 equilateral Δs; (ii) 4 equilateral Δs and 1 hexagon: (iii) 3 equilateral Δs and 2 squares; (iv) 1 equilateral Δ, 1 hexagon and 2 squares; (v) 2 equilateral Δs and 2 hexagons; (vi) 2 equilateral Δs, 1 square, and 1 dodecagon (12-sided figure); (vii) 4 squares; (viii) 1 square and 2 octagons; (ix) 3 hexagons; (x) 1 equilateral Δ and 2 dodecagons; (xi) 1 square, 1 hexagon and 1 dodecagon.

Illustrate, with diagrams, each of the above ways.

12. Show that it is not possible to cover the plane surface about a point with four or more non-overlapping regular polygons of different kinds.

13. How many combinations of three regular polygons (of different kinds) on equal bases are there, that will exactly fill the plane surface about a point? How many of the above combinations permit repetition and continuation in the same plane? Illustrate your answer with accurate figures.

14. Make a table showing the magnitudes of interior and exterior angles of regular figures of 3, 4, ... 15 sides.

Draw a graph to illustrate the relation between (i) the number of sides and the number of degrees in an interior angle; (ii) the number of sides and the number of degrees in an exterior angle; (iii) the number of degrees in an interior angle and an exterior angle of the same polygon.

What inferences can you draw from these graphs?

15. Five angles of a polygon together make up 876°. Each of the remaining angles is 124°. Find the number of sides in the polygon.

16. Show that a rectangle is the only convex rectilineal figure which can have all its angles right.

17. Show that the sum of any p angles of a convex polygon of more than p sides is greater than $(2p - 4)$ right angles.

18. Tabulate all the possible distributions of acute, obtuse, and right angles in a polygon of n sides.

19. Show that no convex polygon can have more than three acute angles.

20 Construct a regular polygon of 9 sides, using your protractor. The side of the polygon may be taken to be 1″.

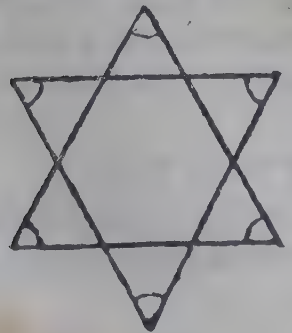


FIG. 66.

21. The alternate sides of a hexagon are produced to meet so as to form a star-shaped figure. Show that the sum of the angles at the vertices of the star is equal to 4 right angles.

Generalise your result for a star with n vertices.

CHAPTER V.

RELATION BETWEEN THE SIDES AND THE ANGLES OF A TRIANGLE.

§ 1. Definitions.

If two sides of a triangle are equal, the triangle is said to be **isosceles**; the angle between the equal sides is called the **vertical angle** and the third side, the **base**.

If no two sides of a triangle are equal, the triangle is called **scalene.**

§ 2. Theorem 8.

If two sides of a triangle are equal, the angles opposite to these sides are equal.

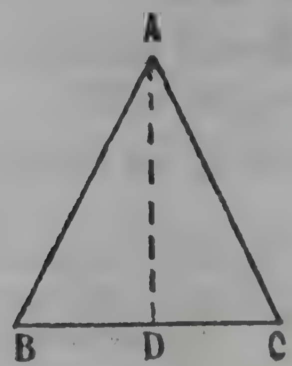


FIG. 67.

Let ABC be a triangle, in which AB = AC.

It is required to prove that $\angle$ACB = $\angle$ABC

Hypothetical Construction: Let AD bisecting the angle BAC meet BC in D.

Proof: Fold the triangle ABC about AD.

Since $\angle$DAC = $\angle$DAB, AC will fall on AB, and since AC = AB, C will coincide with B.

Hence DC must coincide with DB, for the two st. lines DC, DB cannot enclose a space.

$\therefore$ $\angle ACD$ coincides with $\angle ABD$.

$\therefore$ $\angle ACD = \angle ABD$, *i.e.*, $\angle ACB = \angle ABC$.

Cor. 1. Since the two triangles ADB, ADC completely coincide with one another, DB = DC and $\angle ADB = \angle ADC \doteq 90°$.

Thus, *the bisector of the vertical angle of an isosceles triangle also bisects the base and is perpendicular to it.* Conversely, it follows, by the Rule of Identity, that *the perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.*

Cor. 2. If AB, AC be produced, the exterior angles at the base BC are also equal.

Cor. 3. An equilateral triangle is also equiangular. Each of the equal angles is 60°.

Cor. 4. The angles at the base of an isosceles triangle are always acute.

HISTORICAL NOTE. According to Proclus, the discoverer of this theorem was Thales. This proposition has long been known as the *pons asinorum* or the bridge of asses. It is generally surmised that the name arose from the fact that the beginners found this proposition the first stumbling block in Euclid's sequence of theorems. It has also been suggested that the name came from Euclid's figure for this theorem which is in the form of a simple wooden truss bridge (Fig. 68). (For Euclid's as well as Pappus' proof of this theorem, *vide* Chap. VI).

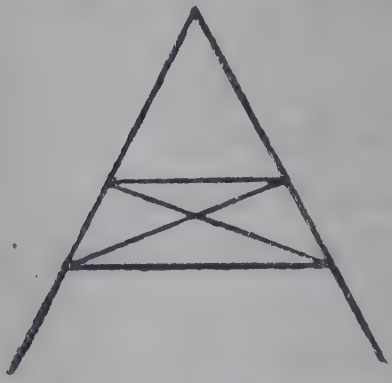


FIG. 68.

EXERCISE X.

1. If in the figure of Th. 8, $\angle ABC = 50°$, what is the magnitude of $\angle BAC$?

2. The vertical angle of an isosceles triangle is 70°. What is the magnitude of an exterior angle at the base?

3. One of the exterior angles at the base of an isosceles triangle is four times the vertical angle. Find the angles of the triangle.

4. One angle of an isosceles triangle is 50°. Find the other angles and show that there are two solutions.

5. The sum of two angles of an isosceles triangle is 100°. Find the magnitude of the vertical angle.

6. If in a quadrilateral ABCD, AB = BC and CD = DA show that $\angle BAD = \angle BCD$.

7. Prove that in a rhombus the opposite pairs of angles are equal.

Def. A **rhombus** is a quadrilateral in which all the sides are equal.

8. O is a point within an acute-angled triangle ABC such that OA, OB, OC are all equal. Prove that $\angle AOB = 2\angle ACB$, $\angle BOC = 2\angle BAC$ and $\angle COA = 2\angle CBA$.

9. In a triangle ABC, O is the middle point of BC and OB = OA. Prove that $\angle BAC = 90°$.

Hence, deduce that the angle in a semi-circle is a right angle.

10. ABC is an isosceles triangle in which AB = AC. D, E are points in AB, AC respectively such that AD = AE. Prove that DE is parallel to BC.

Hence devise a construction for drawing through a given point, a straight line parallel to a given straight line.

11. Show that it is not possible to draw from a point three equal straight lines to meet a given straight line.

12. O is the centre of a circle ABC. P is any point in the arc ACB (vide fig. 69.) Prove that $\angle AOB = 2\angle APB$.

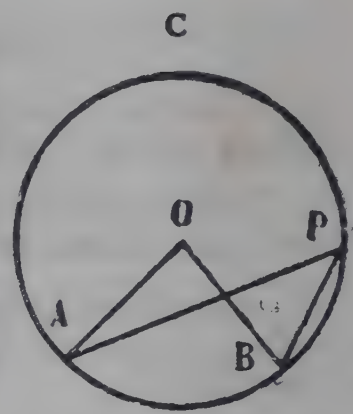


FIG. 69.

Examine whether the result is true for all positions of P on the circle.

13. Assuming that a circle can pass through the vertices of a regular polygon (say, of 5 sides), prove that the radii to the several vertices of the polygon bisect the interior angles. Prove also that the sides of the polygon subtend equal angles at the centre. Hence calculate the magnitude of each of these angles.

14. ABC is an isosceles triangle in which AB = AC. A straight line through B meets the opposite side at D and divides the triangle ABC into two other isosceles triangles. Find the magnitudes of the angles of the triangle ABC.

(Prove that either BD = AD = BC or BD = AD and BC = CD).

15. ABC, ABD are two triangles in which $\angle ABC = \angle ABD$ and AC = AD. If BC is not equal to BD prove that $\angle ACB + \angle ADB = 180°$.

16. O is a point within a quadrilateral ABCD such that OA = OB = OC = OD. Prove that the opposite angles of the quadrilateral are supplementary.

17. ABC is a triangle in which AB > AC. From AB, a length AD is cut off equal to AC, and CD is joined. Prove that $\angle BCD = \frac{1}{2}(\angle ABC \sim \angle ACB)$.

18. On the same base, and on the same side of it, prove that there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another and in the same way those terminated at the other extremity also equal.

§ 3. Theorem 9.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

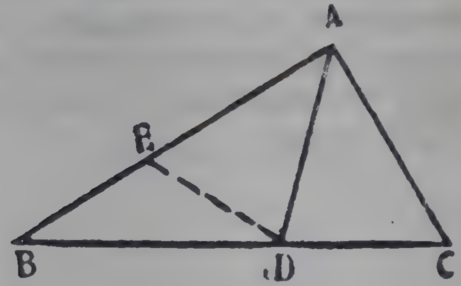


FIG. 70.

Let ABC be a triangle in which AB > AC.

It is required to prove that ∠ ACB > ∠ ABC.

Hypothetical Construction: Let AD bisecting ∠ BAC meet BC in D.

Proof: Fold the triangle ABC about AD.

Since ∠ DAC = ∠ DAB, AC will fall on AB and since AC < AB, C will fall on some point E between A and B and the triangle ADC will occupy the position ADE (as in fig. 70) so that ∠ACB = ∠DEA.

Since DE falls within the angle ADB,

$$\angle ADB > \angle ADE.$$

Now ∠ ADE + ∠ DEA + ∠ EAD = 2 rt. ∠s
= ∠ ADB + ∠ DBA + ∠ BAD (angles of a triangle).

But ∠ EAD = ∠ BAD

and ∠ ADE < ∠ ADB.

∴ ∠ DEA > ∠ DBA or ∠ ABC.

Since ∠ ACB = ∠ DEA,

∴ ∠ ACB > ∠ ABC.

Cor. 1. If two sides of a triangle are unequal, the lesser side always subtends an acute angle at the opposite vertex.

Cor. 2. The angle opposite to the greatest side in a triangle is the greatest angle. Therefore, the angles adjacent to the greatest side must be lesser angles and hence acute.

EXERCISE XI.

1. In a triangle ABC, arrange the angles in order of magnitude; given

(i) $AB = 5''$, $BC = 7''$, $AC = 3''$.

(ii) $AB = 5$ cm., $BC = 2''$, $AC = 7$ cm.

(iii) $AB = BC > AC$.

(iv) $AB + BC = 2AC$; $BC + AC = 3AB$.

2. ABCD is a quadrilateral in which $AB > AD$ and $BC > CD$. Prove that $\angle ADC > \angle ABC$.

3. In the figure of Th. 9, prove that $\angle BED > \angle ABC$.

4. Prove Th. 9, by joining EC or by producing AC to F making $AF = AB$ and joining FB.

5. **ABC is a triangle and D is the mid. point of BC. AD is joined. Prove that $\angle BAC$ is obtuse, right, or acute according as AD is less than, equal to, or greater than $\frac{1}{2}$BC.**

§ 4. **Theorem 10. (Converse of Th. 8).**

If two angles of a triangle are equal, the sides opposite to them are also equal.

FIG. 71.

Let ABC be a triangle in which $\angle ABC = \angle ACB$; it is required to prove that $AC = AB$.

Hypothetical Construction:

Let AD be the bisector of $\angle BAC$ meeting BC in D.

Proof: In the two triangles ADB, ADC,

since $\angle BAD = \angle DAC$

and $\angle ABD = \angle ACD$,

$\therefore$ $\angle ADB = \angle ADC$.

$\therefore$ If the triangle ABC be folded about AD, DC will fall along DB and AC along AB and hence the point of intersection of AC and CD will coincide with that of AB and BD.

Otherwise the two st. lines AB, DB will meet in more than one point, which is impossible.

$\therefore$ C coincides with B.

$\therefore$ $AC = AB$.

Cor. 1. $DC = DB$.

Cor. 2. *If the bisector of the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.*

Cor. 3. *An equiangular triangle is also equilateral.*

Practical Applications.

1. *To find the height of a tree (or a building).*

Let F denote the base and T the top of a tree. Walk from F a sufficient distance FB until the top T is seen at an elevation of 45°.

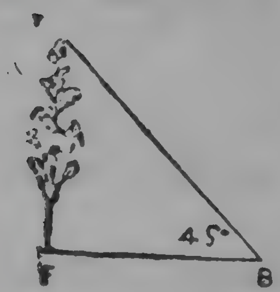


FIG. 72.

Then in the triangle FTB (*vide* fig. 72) $\angle FBT = 45° = \angle FTB$.

$\therefore \quad FT = FB.$

Hence the height of the tree is equal to the distance walked over.

2. *To measure the distance between two points across a river.*

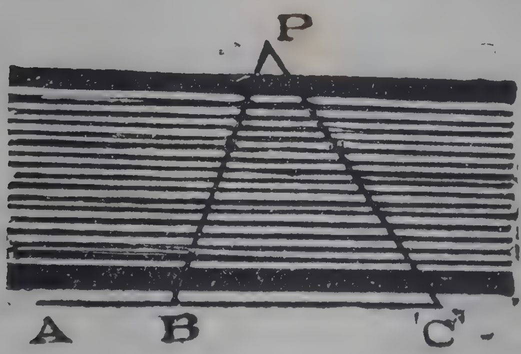


FIG. 73.

Let B, P be two points on opposite sides of a river. Mark a convenient line ABC through B. Observe the angle ABP. Walk along ABC until you reach a point C, such that $\angle ACP = \frac{1}{2} \angle ABP$.

Then since $\angle ABP = \angle BPC + \angle BCP$
and $\angle ABP = 2\angle BCP$.
$\therefore$ $\angle BPC = \angle BCP$
$\therefore$ $BC = BP$.

Hence the distance between the two points is equal to the distance walked over.

N.B. A navigator uses the same method when he *doubles the angle on the bow* to find his distance from a light-house or some other place.

EXERCISE XII.

1. Two angles of a triangle are 50° and 80°. Show that the triangle is isosceles.

2. The difference of two angles of a triangle is 30° and their sum 110°. Show that the triangle is isosceles.

3. If in a quadrilateral, three sides are equal and one pair of opposite angles also equal, show that the quadrilateral is a rhombus and the other pair of opposite angles are also equal.

4. ABC is an isosceles triangle in which $AB = AC$. DE drawn parallel to BC meets AB, AC, in D and E respectively. Prove that $AD = AE$.

5. The bisectors of the angles ABC, ACB of a triangle meet at I. If $IB = IC$, prove that $AB = AC$.

6. The internal (or external) bisectors of the angles ABC, ACB of a triangle meet at I. Through I, a st. line DIE is drawn parallel to BC to meet AB, AC in D, E respectively. Prove that $DE = BD + CE$.

7. The internal bisector of the angle ABC and the external bisector of the angle ACB of a triangle meet in I_2. Thro' I_2, a st. line is drawn parallel to BC to cut AB, AC in D, E respectively. Prove that $DE = BD \sim CE$.

8. The external bisector of the vertical angle of a triangle is parallel to the base. Prove that the triangle is isosceles.

9. ABC is a triangle in which $\angle BAC = 20°$ and $\angle ABC = 35°$. The internal bisector of $\angle BAC$ meets BC in D and the external bisector meets BC produced in E. Prove that AD = AE.

10. ABCDE is a regular pentagon. Prove that its diagonals are all equal to one another. If AC, BD intersect at O, prove that AO = AB.

§ 5. Theorem 11. (Converse of Th. 9).

If two angles of a triangle are unequal the side opposite to the greater angle is greater than the side opposite to the lesser.

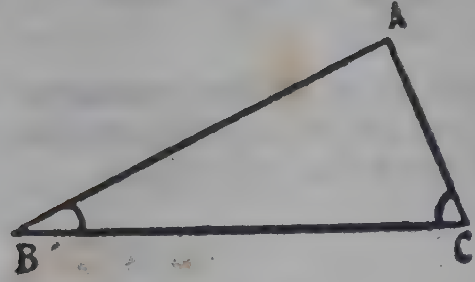


Fig. 74.

Let ABC be a triangle in which $\angle ACB > \angle ABC$.

It is required to prove that $AB > AC$.

Proof: If AB is not greater than AC,

then $AB = AC$ or $AB < AC$.

If $AB = AC$, $\angle ACB = \angle ABC$ which is against the hypothesis.

Again, if $AB < AC$, $\angle ACB < \angle ABC$ which is also opposed to the hypothesis.

$\therefore$ AB is neither equal to nor less than AC,

i.e., $AB > AC$.

Cor. 1. In a scalene triangle, the side opposite to the greatest angle is greatest. Thus, *the hypotenuse* (*i.e.*, the side opposite to the right angle) *is the greatest side in a right-angled triangle.*

In an obtuse-angled triangle the side opposite to the obtuse angle is the greatest side.

Cor. 2. *The perpendicular is the shortest line that can be drawn from a given point to a given st. line.* (The perpendicular distance is also known simply as the distance of the point from the line).

NOTE 1. The method of proof adopted in this theorem is known as the method of '**reductio ad absurdum,**' *i.e.*, the method of disposing of successively all the possible conclusions except one, as inconsistent with the data and leading to an absurdity. This is sometimes called also 'the method of exhaustion.'

NOTE 2. It may be observed here that the proof of this proposition involves a logical principle known as the *Law of Converses.* It may be enunciated thus :

If A can be either A_1, A_2, or A_3 and nothing else and if $A = A_1$, then $B = B_1$; if $A = A_2$, then $B = B_2$; and if $A = A_3$, then $B = B_3$; then the converses of these are also true ; *viz.*

If $B = B_1$ then $A = A_1$ for otherwise $A = A_2$ or A_3 which necessitates $B = B_2$ or B_3 and this is contradicted by the hypothesis $B = B_1$. Similarly, we prove that if $B = B_2$, then $A = A_2$; and if $B = B_3$, then $A = A_3$.

The Law of Converses can be easily taught to the pupils so that they may appreciate its several applications in Geometry in proving not one converse theorem but a group of converses.

By the Law of Converses, Ths. 10 and 11 readily follow from Ths. 8 and 9; for, the latter theorems state: if $AB = AC$, $\angle ACB = \angle ABC$, if $AB > AC$, $\angle ACB > \angle ABC$, if $AB < AC$, $\angle ACB < \angle ABC$ and necessarily AB cannot be anything else than equal to or greater or less than AC; and their converses are therefore true.

For instance, to prove that $AB = AC$ if $\angle ACB = \angle ABC$, we dispose of the cases $AB < AC$ and $AB > AC$ which imply respectively $\angle ACB < \angle ABC$ and $\angle ACB > \angle ABC$ as inconsistent with the given hypothesis; therefore $AB = AC$.

NOTE 3. The proof of the kind explained above is said to be indirect, since the truth of the theorem is made to come out indirectly (by circumstantial evidence, as it were) by showing that the other possible conclusions lead to absurdity. The indirect proof is often deemed to be less worthy than the direct proof. It shows that a theorem *is* true without explaining *why* it is true.

The indirect proof is something like the defence of the 'Alibi' that is pleaded in criminal courts.

NOTE 4. The theorems 10 and 11 may be called the **reciprocals** or **duals** of the two preceding ones 8 and 9.

In fact, the enunciations of Theorems 10 and 11 can be obtained respectively from those of Th. 8 and 9 by interchanging the words 'sides' and 'angles.'

In general, to every proposition involving points and lines, there is a **dual** or **reciprocal** proposition involving lines and points respectively, which is true in a certain type of propositions. This relation is known as the **Principle of Reciprocity or Duality.** In the theorems in question, we have angles instead of points. It is interesting to point out to pupils such dual relations between theorems in Elementary Geometry, which will at least help them in remembering associated theorems.

EXAMPLE 1. *ABC is a triangle in which AB > AC. AD is any straight line from A to a point D in BC. Show that AB > AD.*

Proof:

$\angle$ADB (an ext. $\angle$ of $\triangle$ ADC) > $\angle$ACD (an int. opp. $\angle$).

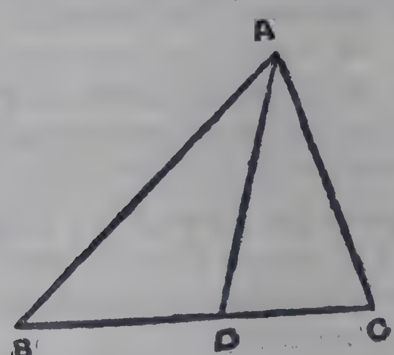


FIG. 75.

Since AB > AC, $\angle$ACB, or $\angle$ACD > $\angle$ABC.

$\therefore$ $\angle$ADB > $\angle$ABC or $\angle$ABD.

Hence in the $\triangle$ ABD, the side AB opposite to the greater angle ADB is greater than the side AD opposite to the lesser angle ABD.

Cor. Any straight line drawn within the triangle ABC is less than the greatest side of the triangle

EXAMPLE 2. *In a right-angled triangle, prove that the straight line joining the mid. point of the hypotenuse to the opposite vertex is equal to half the hypotenuse.*

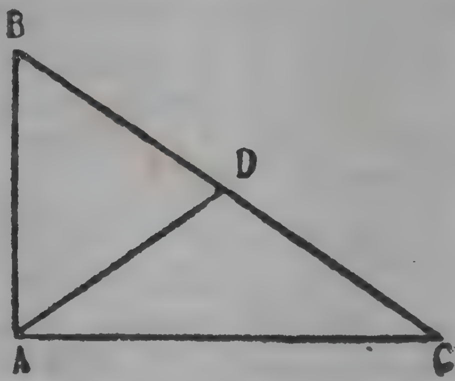


FIG. 76.

Let D be the middle point of the hypotenuse BC of the right-angled triangle ABC. It is required to prove that AD = $\frac{1}{2}$BC.

If AD $\neq \frac{1}{2}$BC, then AD > $\frac{1}{2}$BC or AD < $\frac{1}{2}$ BC.

If AD > $\frac{1}{2}$BC, *i.e.*, DB or DC, then $\angle$ABD > $\angle$DAB and $\angle$ACD > $\angle$DAC.

$\therefore$ $\angle$ABD + $\angle$ACD > $\angle$DAB + $\angle$DAC,

i.e., > $\angle$BAC.

To each add $\angle$BAC.

Then $\angle$ABD + $\angle$ACD + $\angle$BAC > 2 $\angle$BAC,

i.e., 2 rt. $\angle$s > 2 $\angle$BAC, *i.e.*, > 2 rt. $\angle$s

which is absurd. $\therefore$ AD $\ngtr \frac{1}{2}$BC.

Similarly, it can be shown that AD $\nless \frac{1}{2}$BC.

$\therefore$ AD = $\frac{1}{2}$BC.

EXERCISE XIII.

1. Two angles of a triangle are 50° and 30°. Which is the greatest side in the triangle?

2 Without measuring any side, how will you determine which is the greatest side in a triangle by means of protractor?

3. BC is the greatest side of a triangle ABC. Prove that the perpendicular AD from A to BC falls within the triangle. If D is the foot of the perpendicular, prove that AB $>$ BD and AC $>$ CD.

Hence deduce that the sum of the two lesser sides of a triangle is greater than the greatest side.

4. ABC is a triangle in which A $= 80^\circ$, B $= 50^\circ$. A line AD is drawn to meet BC at D such that $\angle$CAD $= 15^\circ$. Show that AB $=$ BD, AC $>$ CD, and AB $+$ AC $>$ BC.

5. O is a point within a triangle ABC such that OB $=$ AB. Prove that AC $>$ OC.

6. ABC is a triangle. The bisectors of the exterior $\angle$s at B and C meet at D. If AB $<$ AC, prove that BD $>$ CD. What is the corresponding result for the internal bisectors of the angles B and C?

7. The side BA of a triangle ABC is produced to D so that AD $=$ AC. Prove that BD $>$ BC.

8. ABC is a triangle. The bisector of the angle ABC meets AC at E and the external bisector of the angle A at F. Prove that AE $<$ EF.

9. ABC is a triangle obtuse-angled at A and D is the middle point of BC. Prove that AD $< \frac{1}{2}$BC.

Enunciate and prove the corresponding theorem when $\angle$BAC is an acute angle.

10. The bisector of the exterior angle at A of a triangle ABC meets BC produced at D. Prove that AB $>$ AC.

11. Show that the perimeter of a triangle is greater than the sum of its altitudes.

12. ABCD is a quadrilateral in which $\angle$ABC $=$ $\angle$CDA $= 90^\circ$. If O is the middle point of AC, prove that OA $=$ OB $=$ OC $=$ OD.

§ 6. Theorem 12.

The sum of any two sides of a triangle is greater than the third side

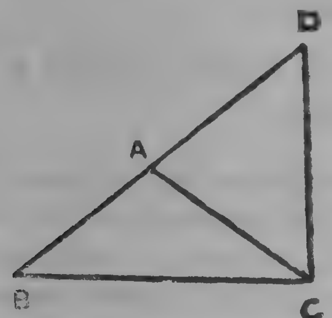


FIG. 77.

Let ABC be any triangle. It is required to prove that $AB+AC>BC$, $BC+CA>BA$, and $AB+BC>AC$.

Case (i). If $AB=AC=BC$, the truth of the theorem is obvious.

Case (ii) If the sides are unequal, let BC be the greatest side. Then it is obvious that $BC + CA > BA$ and $BC+AB>AC$.

Thus the only case that remains to be proved is that $AB+AC>BC$.

Construction: Produce BA to D making $AD=AC$. Join DC.

Proof: Since $AD = AC$

$\therefore\quad \angle ACD = \angle ADC$ or $\angle BDC$.

Evidently $\angle BCD$ is greater than its part $\angle ACD$

$\therefore\quad \angle BCD > \angle BDC$

In the triangle BCD, since $\angle BCD > \angle BDC$

the side $BD >$ the side BC

i.e. $BA+AC>BC$.

Cor. 1. The difference of two sides of a triangle is less than the third side.

Cor. 2. Each side of a triangle is less than half the perimeter of the triangle.

Cor. 3. The sum of two sides of a triangle is greater than half the perimeter.

NOTE: This theorem enunciates the most vital condition necessary and sufficient for the existence and the construction of a triangle with given sides. It states a fundamental relation between the sides of a triangle just in the same way as Th. 6 which states a fundamental relation between the angles, but differs from the latter in one respect. While Th. 6 is not necessarily true, being deduced from Playfair's axiom which may well be replaced by axioms of opposite import, Th. 12 is necessarily and also absolutely true and so is the first proposition that gives a genuine and absolute property of the triangle.

In one view, however, this theorem may be considered as a corollary of the axiom:

The straight line is the shortest distance between two points.

Thus the truth of this theorem is semi-obvious, unlike that of Th. 6.

It is said that Epicureans, the followers of Epicurus, the Greek philosopher of the 4th century B C., used to ridicule this theorem on the score that even an ass knew it; for to take the food placed at a distance from him, he would walk straight towards it and not along two sides of a triangle. But knowing a theorem to be true is different from knowing why it is true. It is well to bear in mind that the value of geometry lies rather in the proof than in the facts.

Example. O is a point within a triangle ABC. Prove that OB+OC<AB+AC.

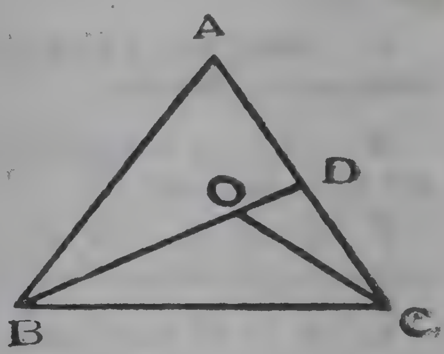


FIG. 78.

Construction: Produce BO to cut AC at D.

Proof: In the ΔABD, AB+AD>BD

i.e. >OB+OD

Again, in the ΔODC, OD+DC>OC.

Adding, AB + AD + OD +DC>OB+OD+OC.

Taking away OD from both sides,
AB+ AD + DC > OB + OC
i.e. AB + AC > OB + OC.

Cor: Similarly AB+BC>OA+OC
and AC+BC>OA+OB
$\therefore$ 2 (AB+AC+BC) > 2 (OA+OB+OC)
$\therefore$ AB+AC+BC > OA+OB+OC

Hence, *the sum of the distances of a point within a triangle from the vertices is less than the perimeter of the triangle.*

EXERCISE XIV.

1 Two sides of a triangle are 5″ and 4″. Find the limits for the length of the third side.

2. Is a triangle possible with the following data :

(i) AB=5″, AC=6 cms, BC=7·3″

(ii) AB+AC=3″, AC+BC=4″, AB+BC=5″?

3. Show that a triangle is always possible with the following lengths for sides (a, b, c being all positive):—

(i) $a+b$, $b+c$, $c+a$.

(ii) $a+2b+c$, $b+2c+a$, $c+2a+b$.

4. Prove Th. 12 either by bisecting one of the angles, or by cutting off from the greater side a part equal to the less.

5. Prove that a diameter of a circle is greater than any chord not a diameter.

6. ABC is a triangle in which the angles at B and C are each less than 60°. Prove that $AB+AC<2BC$.

7. Show that it is not possible to have a triangle with sides a, b, $2c$ when $c>a>b$.

8. If O be any point in the plane of a quadrilateral ABCD, prove that $OA+OB+OC+OD \geqq AC+BD$. For what position of O is $OA+OB+OC+OD$ least?

9. Prove that the sum of the diagonals of a quadrilateral is greater than the semi-perimeter but less than the perimeter of the quadrilateral.

10. Show that the perimeter of any triangle entirely within another is less than the perimeter of the latter.

11. O is any point within an equilateral triangle ABC. Prove that the sum of any two of the three lengths OA, OB, OC is greater than the third. Hence show that it is possible to construct a triangle with OA, OB, OC as sides.

12. If from any point within a circle which is not the centre, straight lines are drawn to the circumference, prove that the greatest is that which passes through the centre and the least the remaining part of the diameter.

13. If from any point without a circle straight lines are drawn to the circumference, of those which fall on the concave circumference, prove that the greatest is that which passes through the centre; and of those which fall on the convex circumference, prove that the least is that which when produced passes through the centre.

14. Find the greatest and the least straight lines which have their extremities one on each of two given non-intersecting circles.

CHAPTER VI.

CONGRUENT TRIANGLES.

§ 1. Definitions and Preliminary Remarks.

Notation: The sides BC, CA, AB of a triangle ABC are denoted by the small letters a, b, c and the angles at the opposite vertices by the corresponding capitals A, B, C.

Def. The three sides and the three angles of a triangle are called the six *elements* of the triangle.

Two triangles ABC, A′B′C′ are said to be **congruent** or **equal in all respects** or **identically equal** when the six elements of one triangle are severally equal to the **corresponding*** six elements of the other triangle.

Symbolically, the congruence is expressed thus :

$$\triangle ABC \equiv \triangle A'B'C'$$

where the order of the letters follows the order of the equal pairs of angles†.

But it can be shown that if *three* out of the *six* elements of one triangle be equal to the corresponding *three* elements of another triangle, then the triangles are congruent, provided the three elements are not angles, nor two elements sides and the third an angle opposite to one of these elements.

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* **Corresponding** sides are those which are opposite to the equal angles.

† This arrangement is convenient as it enables us to read mechanically the corresponding sides without consulting any figures.

Thus two $\triangle$s ABC, $A'B'C'$ are congruent, if one of the following three sets of conditions holds:—

(i) $a=a'$, $b=b'$, $C=C'$

(ii) $A=A'$, $B=B'$, $c=c'$ or $a=a'$

(iii) $a=a'$, $b=b'$, $c=c'$

These cases of congruence can be easily inferred from the fact that we can construct *only one* triangle with the given parts

$$a,\ b,\ C;\ A,\ B,\ c \text{ or } a;\ a,\ b,\ c.$$

Conversely, the congruence theorems assert that wherever, whenever, and in whatever manner we construct triangles with these parts, the triangles so obtained are identically equal.

§ 2. Theorem 13.

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

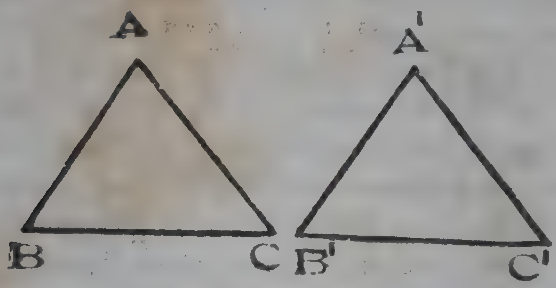

FIG. 79.

Let ABC, A′B′C′ be two triangles in which

$$AB = A'B'$$
$$AC = A'C'$$
$$\angle BAC = \angle B'A'C'$$

It is required to prove that $\triangle ABC \equiv \triangle A'B'C'$
i.e. $BC = B'C'$, $\angle ABC = \angle A'B'C'$
and $\angle ACB = \angle A'C'B'$.

Proof: Apply the triangle A′B′C′ to the triangle ABC so that A′ falls on A and A′B′ along AB. Then, since $A'B' = AB$, B′ falls on B.

Again, since $\angle B'A'C' = \angle BAC$, A′C′ must fall on AC; and since $A'C' = AC$. C′ coincides with C.

Because B′ coincides with B and C′ with C, the side B′C′ must coincide with BC. Otherwise the two straight lines B′C′ and BC will enclose a space, which is contradictory to the axiom about straight lines.

Thus, the triangle A′B′C′ coincides with the triangle ABC and is identically equal to it; so that $BC = B'C'$, $\angle ABC = \angle A'B'C'$ and $\angle ACB = \angle A'C'B'$.

Cor: ΔABC is equal in area to $\Delta A'B'C'$.

A Note on Superposition and the Principle of Congruence.

The proof given above is due to Euclid and is based on the method of superposition. As a practical test of equality, superposition is invaluable and even indispensable. For, as Prof. H. F. Baker says, 'Is not the test of equal length, for school boys* at least, the possibility of the superposition of the same measuring rod?' But, theoretically, it is open to several objections, of which Euclid himself was probably aware. At any rate as early as the 16th century, we find objections to this method clearly stated by John Peletier in his edition of Euclid. He observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of it. For, we can construct one plane figure congruent to another plane figure by superposing it elsewhere and tracing its boundary. This will be against the dignity of geometry and opposed to its spirit.

Another defect (pointed out by Heath) in Euclid's proof outlined above is the assumption that two straight lines can be made to coincide, if they are equal. What Euclid says is that if two straight lines are made to coincide, then they are equal (*vide* Postulate (i) p. 44). He does not mention the converse assumption noted above.

According to Bertrand Russell, Euclid's proof of the first congruence theorem is a tissue of nonsense.

Veronese argues that since geometry is concerned with empty space which is immovable, it is at least strange to make use of real motion of bodies to prove the properties of immovable space. Again, motion without deformation implies a prior notion of equality of spaces occupied by the moving body in the different stages of its motion; and so to argue that two spaces are equal because they were occupied by the same body at different times is to commit a *petitio principii* or a vicious circle.

The moral for the teacher to draw from the above objections is that Theorem 13 proved above should be regarded more as an axiom whose truth is to be assumed than a theorem which can be proved from more fundamental assumptions. It is, however, difficult for a

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* I may add ' in the commercial world also '.

teacher to make his pupils at the Secondary School stage realise fully the logical implications in the superposition method of proof. Many teachers may agree with Smith in the view that for the purdoses of elementary geometry the matter is hardly worth bringing to the attention of a pupil. But it is a valuable eye-opener to an advanced pupil who can understand that the superposition proof is a pretence at proving the unprovable.

To place the logical treatment of congruence on its real foundations, Prof Nunn enunciates the following principles :—

(i) the (assumed) possibility that a figure, occurring anywhere, may be exactly repeated anywhere else, and

(ii) the (assumed) fact that certain elementary constructions, such as drawing a line through two given points, measuring off a given length along a ray, or setting off an angle of given magnitude can be carried out in only one way.

These two principles fuse into what is called the **Principle of Congruence ; 'If a given geometrical construction can be carried out in only one way, it produces equivalent figures whether carried out here or there.'** For instance, on one side of a given line $AB = c$, it is clearly possible to construct only one triangle ABC, such that $AC = b$ and $\angle BAC$ of given magnitude. It follows that triangles drawn to this specification must be equivalent or congruent wherever they be. The so-called superposition-method of Euclid may be interpreted, in Bertrand Russell's words, as merely the transference of our attention from one figure or set of elements to another. 'Actual superposition which is nominally employed by Euclid, is not required.' Thus, in proving the congruence of the $\triangle$s A'B'C' and ABC in Th. 13, our attention has to be drawn to the fact that the $\triangle$s may be supposed to be constructed with the same specifications, two sides of given lengths and an included angle of given magnitude, in two different places and that these specifications which uniquely determine the triangle in one place must also determine it so in the other. Hence, by the **Principle of Congruence**, the triangles are congruent.

Practical Applications.

1. To find the distance between two inaccessible objects, *i.e.*, the distance which cannot be measured directly.

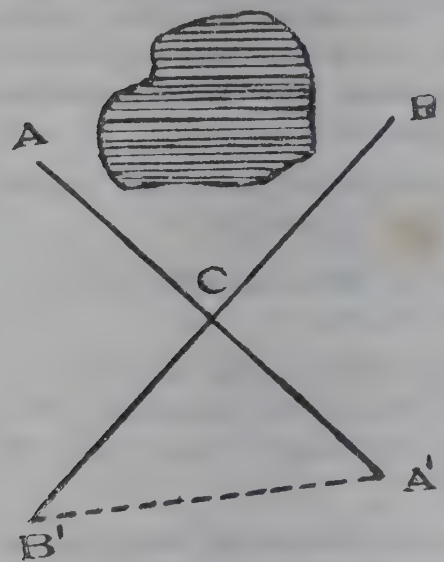

FIG. 80.

Let A, B be two inaccessible objects. Fix a point C from which A and B are both accessible. From C walk in the direction AC produced to A′ so that CA′=CA. Similarly, from C walk in the direction BC produced to B′ such that CB′=CB. Then the distance between A′ and B′ is equal to the distance between A and B; for $\triangle ABC \equiv \triangle A'B'C'$.

2 To construct a rectilineal figure congruent to a given rectilineal figure.

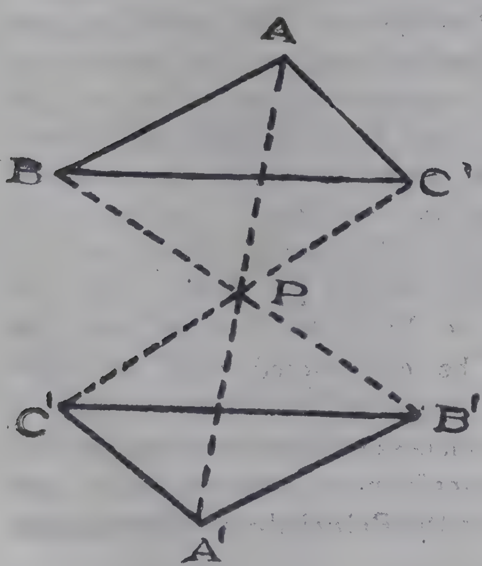

FIG. 81.

Since every rectilineal figure can be divided into triangles, the problem can be made to depend upon the simpler problem of constructing a triangle congruent to a given triangle.

Let ABC be a given triangle To construct another triangle congruent to it, take a point P in the plane of the triangle and join AP,

BP, CP and produce them to A′, B′,C′ respectively, so that PA′, PB′, PC′ may be resepectively equal to PA PB, PC. Join A′B′, B′C′, and C′A′. Then A′B′C′ is the required triangle.

Proof: In the triangles PAB, PA′B′

$$PA = PA'$$
$$PB = PB'$$
$$\angle APB = \angle A'PB' \text{ (vert. opp. angles)}$$
$$\therefore \quad \Delta PAB \equiv \Delta PA'B'$$
so that $AB = A'B'$, and $\angle PAB = \angle PA'B'$.

Similarly, it can be shown that

$$\Delta PAC \equiv \Delta PA'C'$$
so that $AC = A'C'$ and $\angle PAC = \angle PA'C'$

Again in the Δs ABC, A′B′C′

$$AB = A'B' \text{ (proved)}$$
$$AC = A'C' \quad \text{,,}$$
$$\angle BAC = \angle PAB + \angle PAC = \angle PA'B' + \angle PA'C' = \angle B'A'C'$$
$$\therefore \quad \Delta ABC \equiv \Delta A'C'B'.$$

N.B. The above method can obviously be generalised with respect to any rectilineal figure.

EXERCISE XV.

1. Draw a circle of radius 1″. With the help of the protractor, make angles AOB, BOC at the centre O, each equal to 120°, so that the points A, B, C may lie on the circumference.

Join AB, BC, CA.

Prove that $\angle COA = 120°$ and the triangle ABC is equilateral.

2. Draw a circle of radius 10 cm. Draw two perpendicular diameters AOC, BOD meeting at the centre O. Prove that A, B, C, D are the vertices of a square.

3. Prove Theorem 8 using Theorem 13.

4. ABC is a triangle. AC is bisected at D. BD is joined and produced to E so that DE = BD. If CE is joined, show that $\triangle ABD \equiv \triangle CED$.

Hence deduce that the exterior $\angle$ at C is greater than $\angle BAC$.

5. ABC is a triangle. AD is drawn to bisect BC at D. If $\angle BAD = 40^\circ$ and $\angle DAC = 70^\circ$, prove that AB = 2AD.

6. ABCD is a quadrilateral in which AB = BC = CD and $\angle ABC = \angle BCD$. Prove that AC = BD and BC is parallel to AD.

7. ABCD is a square Points E, F, G, H are taken on the sides AB, BC, CD, DA respectively so that AE = BF = CG = DH. Prove that the fig. EFGH is also a square.

Show that the above result can be generalised for any regular polygon.

8. **Show that every point on the perpendicular bisector of a finite straight line is equidistant from its extremities.**

9. ABC is an isosceles $\triangle$ in which AB = AC. D, E are points in AB, AC produced such that BD = CE. BE and CD are joined. Prove that $\triangle ABE \equiv \triangle ACD$ and $\triangle BCD \equiv \triangle CBE$.

Hence deduce the result of Theorem 8.

(This is Euclid's proof of Th. 8).

10. ABC, DEF are two congruent triangles. G, H are points in BC, EF respectively such that BG = EH. Show that AG = DH.

11. **If the diagonals of a quadrilateral bisect each other, show that the opposite sides are equal and parallel.**

12. ABC, DEF are two directly congruent triangles, *i.e.*, the corresponding vertices in the two triangles occur in the same order, clockwise or anti-clockwise.

Prove that the angles between the corresponding pairs of sides are equal.

13. ABC, DEF are two directly congruent triangles in the same plane. If O is a point in the plane such that OA = OD, and OB = OE, prove that OC = OF.

14. AB is a straight line. How will you find a point C in a line with A and B such that CB = AB, without actually producing the straight line?

15. Prove that, in equal circles, chords which subtend equal angles at the centres are equal.

16. Prove that the bisectors of the angles of a regular polygon are concurrent.

17. When our shadows are of the same length at two different times in a day, show that the altitudes of the Sun at those times must be equal.

§ 3. Theorem 14.

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

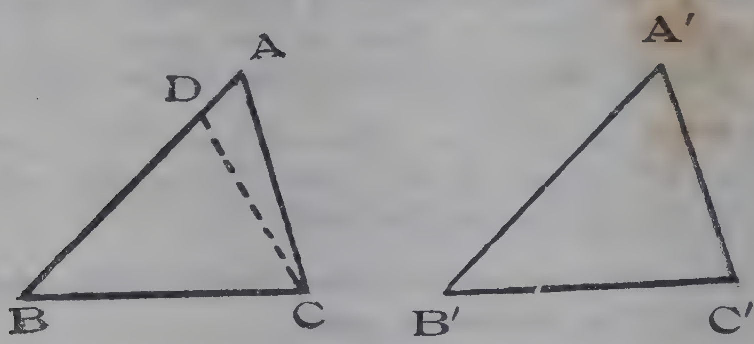

FIG. 82.

Let ABC, A′B′C′ be two triangles in which $\angle ABC = \angle A'B'C'$, $\angle ACB = \angle A'C'B'$.

Since the $\angle$s of each triangle are together equal to two right angles, the third $\angle BAC =$ the third $\angle B'A'C'$. Thus, whatever pair of angles be given to be equal in the two triangles, it follows that the three angles of one triangle are respectively equal to the three angles of the other triangle.*

Also, let $BC = B'C'$ (one pair of corresponding sides).

It is required to prove that $\Delta ABC \equiv \Delta A'B'C'$.

If $BA \neq B'A'$, let $BA > B'A'$. From BA cut off $BD = B'A'$. Join DC.

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* From this it is obvious that this condition of equality of the angles alone is insufficient to establish the congruence of the triangles. We require, in addition, the equality of one pair of corresponding sides, *i.e.*, the pair of sides opposite to a pair of equal $\angle$s, for example $BC = B'C'$ or $CA = C'A'$ or $AB = A'B'$ but not $BC = C'A'$, etc.

Proof: In the Δs DBC, A′B′C′

$\because$ DB = A′B′, BC = B′C′, $\angle$DBC = $\angle$A′B′C′

$\therefore$ ΔDBC $\equiv$ ΔA′B′C′, so that $\angle$DCB = $\angle$A′C′B′.

But, by hypothesis, $\angle$ACB = $\angle$A′C′B′.

$\therefore$ $\angle$DCB = $\angle$ACB, which is impossible unless DC coincides with AC, *i.e.*, D coincides with A.

$\therefore$ AB = A′B′ and ΔABC $\equiv$ ΔA′B′C′.

NOTE 1. This Theorem can be framed from the preceding by reading angles for sides and *vice-versa* in the enunciation and therefore is a reciprocal of Theorem 18.

NOTE 2. This Theorem is attributed to Thales who is supposed to have made use of it in solving the problem of finding the distances of ships at sea.

But we do not know how Thales did this actually.

The following is one of the conjectures as to how Thales may have applied this Theorem :

To find the distance between two points A, B not directly accessible from one another.

From B measure along a straight line BCD at right angles to AB two equal lengths BC, CD. From D draw DE perpendicular to BD on the side of BD opposite to A and find a point E in it which is in a straight line with A and C. Then, by the above theorem, DE=AB.

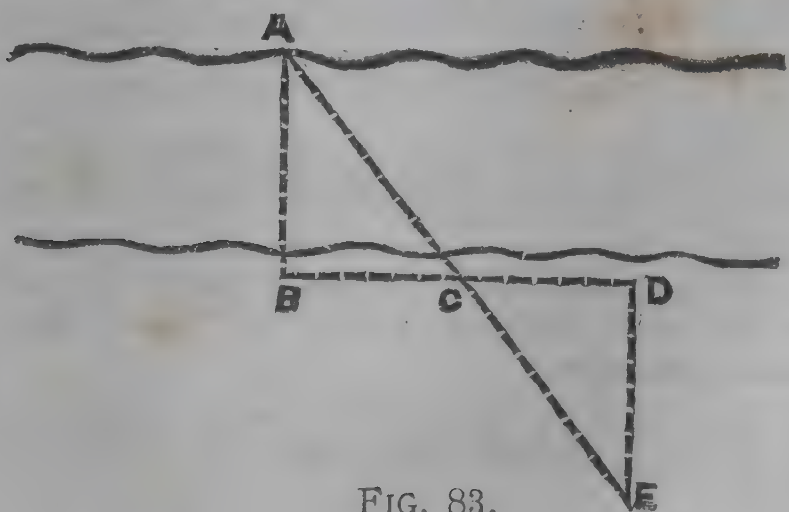

FIG. 83.

Dr. Heath's objection to this method is that it is not quite applicable to the case of the ship, as it requires a certain extent of free and level ground for the construction and measurements. He therefore suggests that the following may have been Thales' method :

'Suppose he was on the top of a tower with a rough instrument made of a straight stick and a cross piece fastened to it capable of turning about the fastening so that it could make any angle with it and remain there at that angle ; to sight a ship, he should only hold the stick vertical and direct the cross-piece towards the ship. Then fixing the cross-piece at the angle thus formed, the stick with the cross-piece rigidly attached to it can be turned round (the stick remaining vertical) to sight an object on the shore, the distance of which from the foot of the tower could be subsequently measured.' By Theorem **14**, the distance of the ship from the tower would be the same as the distance of the object on the shore from the same tower.

EXERCISE XVI.

1. From the top of a tree the angles of depression of two objects in the same horizontal plane through the foot of the tree are observed to be equal. Show that the objects must be equidistant from the foot of the tree.

2. Two ladders inclined at the same angle to the vertical stand on a level ground and just reach the top of a wall. Show that the ladders must be of equal height.

3. Prove Theorem 10 by means of Theorem 14.

4. **Show that every point on the bisector (internal or external) of an angle is equidistant from the arms of the angle.**

5. **If the opposite sides of a quadrilateral are parallel, show that each diagonal divides the figure into two congruent triangles.**

6. **AB is a finite straight line and C its middle point. Show that every straight line through C is equidistant from A and B.**

7. ABC is a triangle. D, E are the middle points of AB, AC respectively. Show that the perpendiculars from A, B, C on DE are equal to one another.

8. ADB, AEC are two straight lines meeting at A. If BE, CD meet at O and AD = AE, BD = EC, prove that

(1) Δ BDO $\equiv$ Δ CEO; (2) Δ AOB $\equiv$ Δ AOC; (3) AO bisects the angle BAC.

Hence deduce a construction for bisecting an angle.

9. If Δ ABC $\equiv$ Δ DEF, prove that the altitudes of the triangle ABC are equal to the corresponding altitudes of the triangle DEF.

10. Prove Theorem 14 directly by superposition.

11. Assuming Theorem 14, deduce Theorem 13 from it by the method of *reductio ad absurdum.*

12. If in two triangles ABC, A′B′C′, the angles B, B′ be equal as also the angles C, C′ and the sides AB, A′B′, prove that the triangles are congruent, without assuming the third angles A, A′ to be equal.

13. If the bisector of the vertical angle of a triangle is perpendicular to the base, show that the triangle is isosceles.

14. ABCD, A′B′C′D′ are two convex quadrilaterals such that AB = A′B′, CD = C′D′, $\angle$A = $\angle$A′, $\angle$B = $\angle$B′ and $\angle$C = C′. Show that the two quadrilaterals are congruent.

15. ABC is a right-angled isosceles triangle having $\angle$ABC = 90°. From A, C perpendiculars AD, CE are drawn to any straight line drawn through B. Prove that DE = AD + CE or AD $\sim$ CE. Distinguish between the two cases.

16. ABC is a triangle. The bisector of the angle BAC meets BC at D. If AB > AC, prove that BD > CD.

§ 4. Theorem 15.

Two triangles are congruent if the three sides of the one are equal to the three sides of the other, each to each.

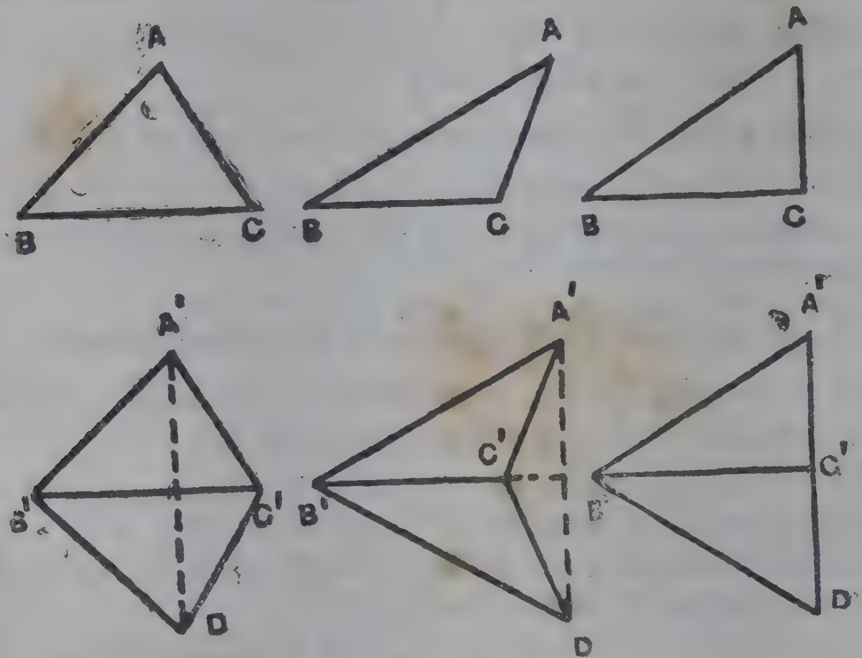

FIG. 84.

Let ABC, A′B′C′ be two triangles in which AB = A′B′, BC = B′C′, AC = A′C′.

It is required to prove that △ ABC ≡ △A′B′C′.

Proof: Place the triangle ABC so that B may fall on B′, and BC along B′C′. Then because BC = B′C′, C will fall on C′. Let the vertex A fall on the side of B′C′ opposite to A′. Let DB′C′ be the new position of the triangle ABC, so that DB′ = AB and DC′ = AC and ∠ B′DC′ = ∠ BAC. Join A′D.

Since A′B′ = AB and AB = DB′,

∴ A′B′ = DB′.

$\therefore \qquad \angle B'DA' = \angle B'A'D.$

Similarly, it can be shown that $\angle C'DA' = \angle C'A'D.$

$\therefore \qquad \angle B'DA' \pm \angle C'DA' = \angle B'A'D \pm \angle C'A'D,$

i.e., $\qquad \angle B'DC' = \angle B'A'C'.$

But $\qquad \angle B'DC' = \angle BAC.$

$\therefore \qquad \angle BAC = \angle B'A'C'$

Hence, in the $\triangle$s ABC, A'B'C'

$$\begin{cases} AB = A'B' & \text{(Given)} \\ BC = B'C' & \text{(Given)} \\ \angle BAC = \angle B'A'C' & \text{(Proved)} \end{cases}$$

$\therefore \qquad \triangle ABC \equiv \triangle A'B'C'.$

HISTORICAL NOTE:—This Theorem is at least as old as Aristotle's time (340 B. C.). The proof given above is not Euclid's but Philo's. Philo of Byzantium lived after Euclid but before the Christian Era.

NOTE 1. In the fig. of Th. 15, we see that there are three cases. But if BC = B'C' be the longest side, then the angles B, C and the corresponding angles B', C' will all be acute and A'D will fall within the quadrilateral A'B'DC'. Thus, only one case need be considered.

NOTE 2. One of the most important applications of this proposition is that a triangular arrangement is always necessary to secure rigidity in any frame-work.

NOTE 3. This theorem has many applications in simple constructions such as, bisecting an angle or a straight line, drawing a perpendicular or parallel to a given straight line, and making an angle equal to a given angle.

(We shall deal with these constructions in a subsequent section.)

EXERCISE XVII.

1. Three rods are hinged together in pairs so as to enclose a triangular figure. Is the figure rigid? Give reasons for your answer.

2. If the legs of a chair are shaky, how will you make them immovable?

3. If four rods are hinged together so as to form a quadrilateral, is its shape definitely fixed?

4. Can you make a rigid frame work in the form of a pentagon with five rods? If not, how many additional rods will you require and how will you place them?

5. ABC is a rigid frame work with AB = AC. D is the middle point of BC. If the plumb-line from A passes through D, show that BC must be horizontal.

6. **Prove that in equal circles equal chords subtend equal angles at the centres.**

7. **If the opposite sides of a quadrilateral are equal, prove that the figure is a parallelogram.**

8. **Two circles cut at X and Y. Prove that XY is bisected at right angles by the line of centres.**

9. **Prove that a diameter bisecting a chord of a circle is perpendicular to it.**

10. **Show that equal chords of a circle are equidistant from the centre.**

11. If in a quadrilateral three sides and three angles are equal, prove that the figure is a square.

12. If any set of three angles of an equilateral pentagon be equal, show that the pentagon is regular.

13. **Prove that the perpendicular bisectors of the sides of a triangle are concurrent, i.e. meet in a point.**

14. **Show that the perpendicular bisectors of the sides of a regular polygon are concurrent.**

15. A, B, C are three points in a circle; and A′, B′, C′ are the points diametrically opposite to A, B, C respectively. Prove that $\Delta ABC \equiv \Delta A'B'C'$.

16. ABCD, A′B′C′D′ are two convex quadrilaterals in which $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and $\angle BAD = \angle B'A'D'$ Show that the quadrilaterals are congruent.

17. **ABC, DCB are two congruent triaagles on the same base and on the same side of it. If AC, BD intersect at O, prove that $OA = OD$, $OB = OC$ and $AD \parallel BC$.**

18. **From a point outside a circle, show that two and only two equal straight lines can be drawn to a circle. Show also that these two equal straight lines are equally inclined to the diameter through the point.**

19. Two chords AB, CD of a circle meet in E. If $EB = ED$, prove that $AB = CD$.

20. **ABC is a triangle in which $AB = AC$. Conceive of this as two triangles ABC, ACB and prove from their congruence the result of Th. 8.**

(This proof is due to Pappus of 3rd century A.D., a Greek commentator on Euclid.)

§ 5. Theorem 16.

Two right-angled triangles are congruent, if their hypotenuses are equal and one side of the one is equal to one side of the other.

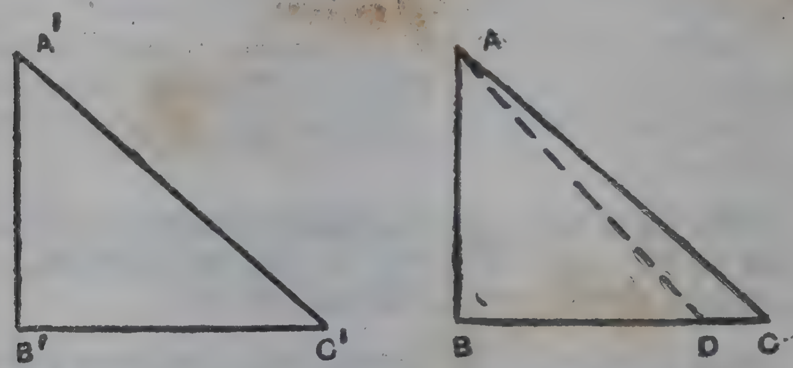

FIG. 85.

Let ABC, A′B′C′ be two triangles in which

$$\angle ABC = 90^\circ = \angle A'B'C'$$
$$AC = A'C'$$
$$AB = A'B'.$$

It is required to prove that $\triangle ABC \equiv \triangle A'B'C'$.

Proof: (i) If $\angle BAC = \angle B'A'C'$, then since $AC = A'C'$, $AB = A'B'$, $\angle BAC = \angle B'A'C'$ (included angles.)

$$\therefore \quad \triangle ABC \equiv \triangle A'B'C'.$$

(ii) If $\angle BAC \neq \angle B'A'C'$, let $\angle BAC > \angle B'A'C'$

At A, let $\angle BAD = \angle B'A'C'$, D being in BC.

In the $\triangle$s ABD, A′B′C′, $AB = A'B'$, $\angle BAD = \angle B'A'C'$ (construction), $\angle ABD = \angle A'B'C'$ (being right angles).

$\therefore \quad \triangle ABD \equiv \triangle A'B'C'$, so that

$$AD = A'C' \text{ and } \angle ADB = \angle A'C'B'.$$

But $\quad A'C' = AC$ (Hypothesis)

$\therefore$ AD = AC.

$\therefore$ $\angle ACD = \angle ADC$ which is impossible since $\angle ACD < 90^\circ$ and $\angle ADC > 90^\circ$.

$\therefore$ AD must coincide with AC.

$\therefore$ $\angle BAC = \angle B'A'C'$.

Hence by (i), $\triangle ABC \equiv \triangle A'B'C'$.

Cor. 1. If $\angle ABC = \angle A'B'C' > 90^\circ$ and $AB = A'B'$, $AC = A'C'$, then $\triangle ABC \equiv \triangle A'B'C'$.

Cor. 2. If $\angle ABC = \angle A'B'C' < 90^\circ$ and $AC > AB$, $A'C' > A'B'$ then $\triangle ABC \equiv \triangle A'B'C'$ provided $AB = A'B'$ and $AC = A'C'$.

Cor. 3. If $\angle ABC = \angle A'B'C' \neq 90^\circ$, and $AB = A'B'$, $AC = A'C'$, then either $\angle ACB = \angle A'C'B'$ and $\triangle ABC \equiv \triangle A'B'C'$ or $\angle ACB = \angle ADC = 180^\circ - \angle ADB = 180^\circ - A'C'B'$. (Vide fig. 86).

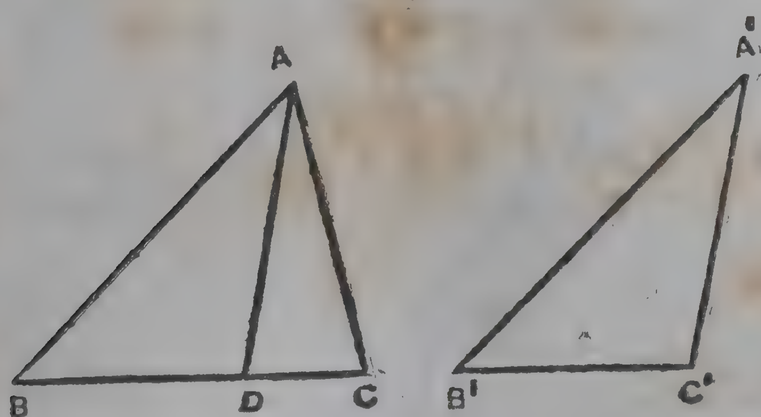

FIG. 86.

It is worth while giving the general enunciation of Cor. 3 above, thus:

If two triangles have two sides of the one equal to two sides of the other, each to each, and if the angles opposite to one pair of equal sides be also equal, then the

angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are congruent.

This is known as the *ambiguous case.*

Alternative Proof of Th. 16.

Place the $\triangle ABC$ so that A falls on A$'$ and AB along A$'$B$'$. Then because AB = A$'$B$'$, B will fall on B$'$. Let the vertex C fall on the side of A$'$B$'$ opposite to C$'$; and let A$'$B$'$D be the new position of $\triangle ABC$.

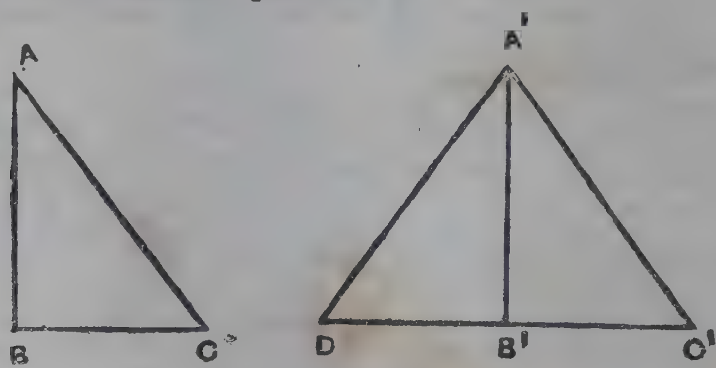

FIG. 87.

Proof: Since $\angle A'B'D = 90°$ and $\angle A'B'C' = 90°$.

$\therefore$ DB$'$C$'$ is a st. line and A$'$DC$'$ is a triangle.

Since $A'D = A'C'$,

$\therefore \quad \angle A'C'D = \angle A'DC'$

i.e. $\angle ACB = \angle A'C'B'$

Hence, in the two $\triangle$s ABC, A$'$B$'$C$'$

$$\angle ABC = 90° = \angle A'B'C'$$
$$\angle ACB = \angle A'C'B' \text{ (proved)}$$
$$AB = A'B'$$

$\therefore \quad \triangle ABC \equiv \triangle A'B'C'$.

NOTE 1. The method of *reductio ad absurdum* or the indirect proof outlined on pp. 120, 121 is applicable to all the cases and therefore educationally more valuable than the alternative proof which has the merit of being 'direct.'

It is interesting to notice, however, that the method of the above alternative proof is useful in proving the following theorem, which can be regarded as a converse of the '*ambiguous case.*'

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides supplementary, then the angles opposite to the other pair of equal sides are equal.

NOTE 2. Another converse of the ambiguous case is the following:

It two triangles have one pair of angles equal and another pair supplementary and the sides opposite to one of these pairs of angles equal, then the sides opposite to the other pair of angles are also equal.

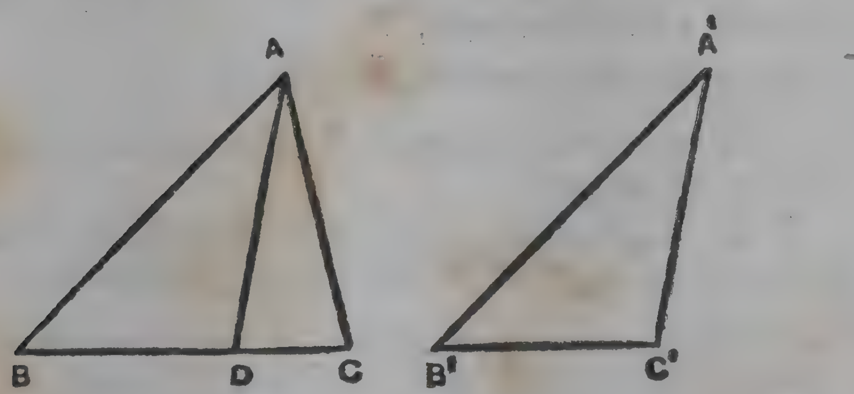

FIG. 88.

Let ABC, A′B′C′ be two Δs in which

$$\angle B = \angle B' \text{ and } \angle C + \angle C' = 180^\circ.$$

Case (i) If $AC = A'C'$, it is required to prove $AB = A'B'$.

The proof of this case is similar to the alternative proof on p. 122.

Case (ii) If $AB = A'B'$, to prove $AC = A'C'$

Of the two angles BAC, $B'A'C'$ one must be greater, for if they are equal, then $\angle C = \angle C' = 90^\circ$ and $\Delta ABC \equiv \Delta A'B'C'$ and $\therefore$ $AC = A'C'$.

Let $\angle BAC > \angle B'A'C'$. Draw AD to meet BC in D so that $\angle BAD = \angle B'A'C'$.

Then, it is easily seen that $\Delta ABD \equiv \Delta A'B'C'$, so that $AD = A'C'$ and $\angle ADB = \angle A'C'B'$

$$\therefore \quad \angle ADC = 180^\circ - \angle ADB = 180^\circ - \angle C' = \angle ACD.$$

$$\therefore \quad AC = AD = A'C'.$$

EXERCISE XVIII.

1. Prove that a diameter of a circle bisects every chord perpendicular to it.

2. Show that chords which are equi-distant from the centre of a circle are equal.

Prove also the converse of the above.

3. If two chords of a circle bisect each other, show that they are diameters.

4. If the bisector of the vertical angle of a triangle bisects the base, show that the triangle is isosceles.

5. If two acute-angled triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides equal, show that the triangles are congruent. Is this theorem true for (i) obtuse-angled Δs, (ii) right-angled Δs?

6 ABC, DEF are two triangles in which $AB = DE$ and $BC = EF$ and $\angle ACB = \angle DFE$; are the triangles congruent if $\angle BAC$ is obtuse?

7. ABC is an equilateral triangle. On BC a square $BCED$ is described in any manner. AD cuts BE at O. Prove that $\Delta AOC \equiv \Delta EOC$.

8. ABCD is a quadrilateral in which a pair of opposite angles are supplementary. If AC bisects the angle BAD, show that $BC = CD$.

9. ABC is a triangle in which $AB > AC$. The perpendicular bisector of BC and the internal bisector of $\angle BAC$ meet in D. Prove that $\angle ABD$ is supplementary to $\angle ACD$. What is the corresponding result for the case of the external bisector?

10. ABC is any triangle. D, E are points in AB, AC such that $BD = CE$. DF, EG are drawn parallel to the bisector of the angle BAC to meet BC iu F, G respectively. Prove that $BF = CG$.

Hence, if AX bisecting the angle BAC cuts BC at X, and $AB > AC$, prove that $BX > CX$ and conversely.

§ 6. Constructions based on Congruent Triangles.

Construction 1.

To bisect a given finite straight line.

Construction 2.

To draw the perpendicular bisector of a given finite straight line.

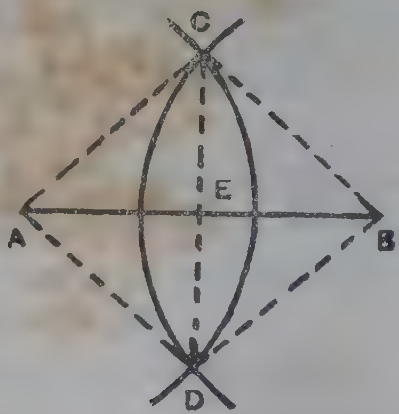

FIG. 89.

Construction. Let AB be the given st. line to be bisected. With centres A and B and any convenient radii draw arcs to cut one another in C and D.

Join CD and let it cut AB at E

Then AB is bisected at E and CD is the perpendicular bisector of AB.

Proof: Join AC, CB, BD, DA.

In the $\triangle$s ACD, BCD,

$$\begin{cases} AC = BC & \text{(radii of equal circles)} \\ AD = BD & \text{"} \\ CD \text{ is common.} & \end{cases}$$

$\therefore$ $\triangle ACD \equiv \triangle BCD$, so that

$\angle ACD = \angle BCD$ *i.e.* $\angle ACE = \angle BCE$

Again in the $\triangle$s ACE, BCE

$$\begin{cases} AC = BC \\ CE \text{ is common} \\ \angle ACE = \angle BCE \quad \text{(proved)} \end{cases}$$

$\therefore \quad \triangle ACE \equiv \triangle BCE$, so that

$$AE = BE$$

$$\angle AEC = \angle BEC$$

Since these are adjacent angles, each of them is right angle.

$\therefore$ E is the mid. pt. of AB and CD is the perpendicular bisector of AB.

HISTORICAL NOTE. The construction outlined above was quite familiar among ancient Indian mathematicians, who called it the construction of the fish-shaped figure* (*Matsyakriti*). The two arcs forming a lune resembling the body of a fish must have suggested the name to them. They used the construction to determine the North-South line by drawing a perpendicular to the East-West line; the latter was obtained by observing the direction of the *shadow*† of a gnomon at the beginning of a day. In this connection it is interesting to note what Proclus says, *viz.* the problem of drawing a perpendicular from a given point to a straight line was first investigated by Oenopides (5th century B. C.) who thought it useful for astronomy.

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* *Vide* Pp. 152 and 275 Mahaviracharya's *Ganitasara Sangraha*, translated by M. Rangacharya (1912).

† More particularly, the equinoctial shadow, *i. e.*, the shadow on the day when the Sun is on the equator and the night and day are equal in length, *i.e.*, about 21st March and 23rd September.

EXERCISE XIX.

1. Draw a straight line equal in length to your ruler and divide it into (i) 2, (ii) 4, and (iii) 8 equal parts.

2. In fig. 89, how will you take the radii so as to ensure the circles cutting?

3. Suppose AB is so long that the compasses are too short to draw circles (with centres A, B) which can intersect, how will you find the middle point of AB?

4. Draw an obtuse-angled triangle. Draw the perpendicular bisectors of its sides Where do they meet? Repeat the process with a right-angled triangle and an acute-angled triangle. Note the difference in the positions of the common intersection of the perpendicular bisectors.

Show that this common intersection is equidistant from the vertices of the triangle. Hence, draw the circum-circle (*i. e.*, the circle passing through the vertices) of the triangle.

5 Draw a quadrilateral ABCD in which the angles ABC and ADC are supplementary. Draw the perpendicular bisectors of the sides and diagonals. Do they all pass through the same point? If so, draw the circum-circle of the quadrilateral.

6. Draw a chord of a circle. Draw the perpendicular bisector of the chord and bisect the portion of the perpendicular bisector intercepted by the circle. Which point do you get by this construction?

7. Draw any quadrilateral ABCD. Bisect AB and CD at E and F respectively. Join EF and bisect it at G. Similarly, bisect AD, BC at E', F' respectively and take G' the mid. point of $E'F'$; and bisect AC, BD at E'', F'' and take G'' the mid. point of $E''F''$. What do you notice with regard to the points G, G', G''?

8. Verify the following constructions for bisecting a straight line AB :—

(i) On AB construct any triangle ACB. With centres A and B and radii equal to BC and AC respectively, draw arcs to cut at a point D on the side of AB opposite to C. Join CD and let it cut AB at E.

Then E is the mid. point of AB.

(ii) Draw any straight line CD parallel to AB and on it take two equal lengths CE, ED. Join AC, BD and produce them if necessary to meet at F.

Then EF produced bisects AB.

(iii) Through A draw any straight line AC and on it take AO = 2 units and OC = 1 unit. Join BC and produce it to D making CD = BC.

Then DO produced bisects AB.

(iv) Through A, B draw two equal and parallel straight lines AC, BD in opposite directions.

Then CD passes through the mid. point of AB.

(v) With centre A and radius AB draw a circle (say C_1). With centre B and radius BA draw another circle (say C_2) to cut C_1 at O.

On the circumference of C_2 step off OP, PQ successively, each = AB.

With Q as centre and QA as radius draw a circle to cut C_1 at D and E.

With D and E as centres and radii equal to AB draw arcs to cut at a point X on the same side of DE as O.

Then X is the mid. point of AB.

N. B. The construction (v) is effected with compasses only. About 1800 A.D. Lorenzo Mascheroni, Professor of Mathematics at Pavia, showed that the ordinary constructions of Elementary Geometry can be performed by the use of the compasses alone, with no assistance from the ruler.

Construction 3.

To bisect a given angle.

Construction 4.

To bisect a straight angle or to draw a perpendicular to a given straight line from a given point in it.

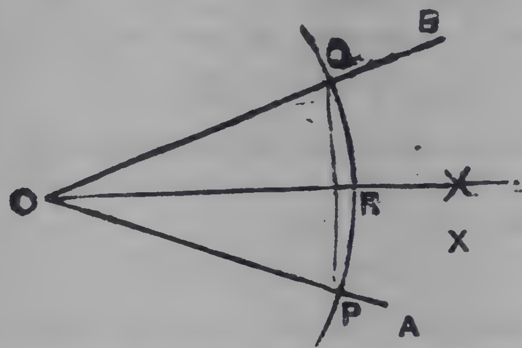

FIG. 90.

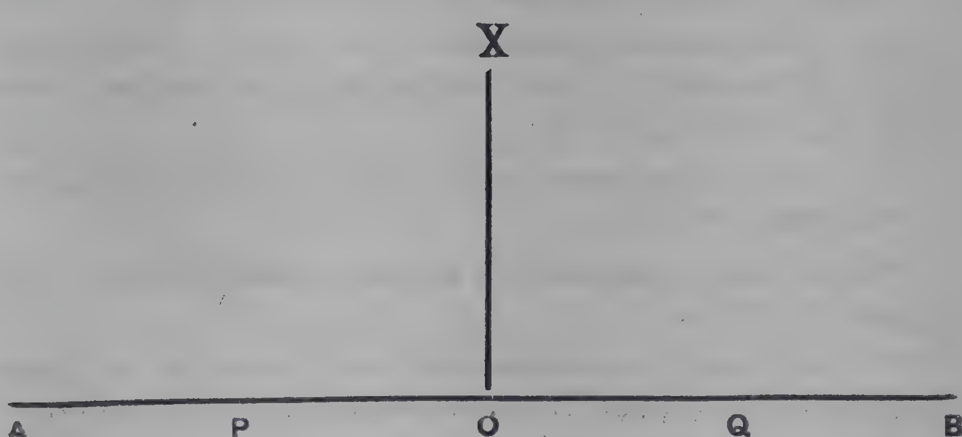

FIG. 91.

Let AOB be the given angle, which may be a straight angle as in Fig. (91).

It is required to bisect the angle.

Construction: With centre O and any radius, draw an arc to cut OA at P and OB at Q. With centre P and any convenient radius draw an arc and again, with centre Q and the same radius, draw another arc to cut the former at X. Join OX, PX and QX (not shown in the figure).

In Fig. (90), OX bisects $\angle$AOB ; in Fig. (91), OX bisects the straight angle AOB, *i.e.*, is perpendicular to the straight line AB at O.

Proof: In the Δs POX, QOX

$$\begin{cases} OP = OQ \\ XP = XQ \\ OX \text{ is common.} \end{cases}$$

$\therefore$ $\Delta POX \equiv \Delta QOX$.

$\therefore$ $\angle POX = \angle QOX$,

i.e., OX bisects $\angle$AOB.

If AOB is a straight line, $\angle$s AOX, BOX are equal adjacent angles (formed by one straight line standing on another) and, therefore, right angles ;

i e. OX is perpendicular to AB at O.

EXERCISE XX.

1. Draw an angle ABC and bisect it by a straight line BD. At B in BD erect BE perpendicular to BD. Show that BE bisects the exterior angle at B.

Check your results by measurement.

2. Show how you will trisect an angle approximately by dividing an angle successively into four equal parts.

For example, take an angle AOB of 40°. Divide it into four equal parts by the lines OC, OD, OE. Again, draw OF such that $\angle COF = \frac{1}{4} \angle COD$ and OF′ such that $\angle EOF' = \frac{1}{4} \angle EOD$.

Calculate the magnitudes of the angles AOF, FOF′ and F′OB.

What is the error in assuming that OF, OF′ trisect the angle AOB?

3. Draw two perpendicular diameters AOA′, BOB′ of a circle. Bisect the angles at O and let the bisectors cut the circumference.

Hence, show how you can inscribe within a circle a regular octagon.

4. With ruler and compasses only, construct angles of the following magnitudes :—15°, 30°, 45°, 60°, 75°, 90°, 105°, 120°, 135°, 150° and 165°.

5. Prove the following construction for getting an angle equal to half of a given angle: Let AOB be the given angle. Produce AO to C. With centre O and radius OC draw an arc to cut OB at D. Then $\angle OCD = \frac{1}{2}\angle AOB$.

Obtain in the same way $\frac{1}{4}\angle AOB$, $\frac{1}{8}\angle AOB$, and so on.

6. Draw any triangle ABC. Bisect the angles ABC and ACB. Let the bisectors meet at I. Verify by actual construction and prove theoretically that the bisector of the angle BAC passes through I.

Also, bisect the exterior angles at B and C and let the bisectors meet at I_1. As before, verify by construction and prove that AI_1 bisects the angle BAC.

Use the fact, that II_1 produced bisects the angle BAC, to devise a construction for the bisector of an angle whose vertex is inaccessible.

7. Verify the following constructions for drawing a perpendicular to a straight line AB at a given point P in it and prove them:

(i) With P as centre and any convenient radius, draw an arc DEF to cut PB at D.

With centre D and the same radius as before draw an arc to cut the former arc at E and again with centre E and the same radius draw an arc FG to cut the arc DEF at F.

Lastly with centre F and the same radius as before draw an arc to cut the arc FG at G on the side of FE opposite to P. Join PG.

Then PG is perpendicular to AB.

(ii) Take any point C outside AB. If CP is perpendicular to AB, there is no need of further construction. If CP is not perpendicular to AB, draw a circle with C as centre and CP as radius to cut AB again at Q and produce QC to cut this circle again at G. Join PG.

Then PG is perpendicular to AB.

N. B. The above constructions do not involve the production of the straight line AB on both sides of P. For the purpose of the constructions, the portion of the straight line on one side of P may be entirely ignored.

8. Show how you will erect a perpendicular at one extremity of a straight line, without producing the straight line beyond that extremity. (Use only ruler and compasses.)

9. Prove the following construction for trisecting a given straight line AB:

At A and B construct angles BAC, ABC each equal to 30°. Draw the perpendicular bisectors of CA, CB to cut AB at D and E.

Then AD = DE = EB.

10. AB is the diameter of a semi-circle ACB. P is a point in AB such that chord AC = AP. If AC is produced to G so that AG = AB, prove that PG is perpendicular to AB.

Hence, devise a construction for erecting a perpendicular to AB at P.

Construction 5.

To draw a straight line perpendicular to a given straight line from a given point outside it.

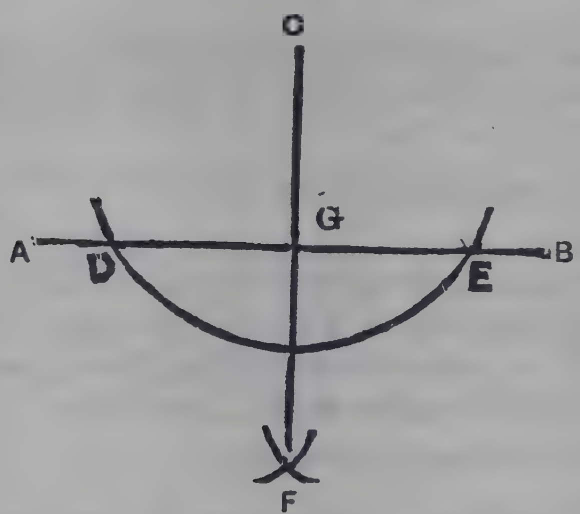

FIG. 92.

Let AB be the given straight line and C a point outside it.

Construction: Take a point D in AB. If CD is perpendicular to AB, no further construction is needed. Otherwise, with centre C and radius CD, draw an arc to cut AB or AB produced at E.

Bisect DE at G. Join CG.

Then CG is perpendicular to AB.

Proof: Join CD, CE.

In the $\triangle$s CGD, CGE,

$$\begin{cases} CD = CE \\ GD = GE \text{ (construction)} \\ \text{and } CG \text{ is common.} \end{cases}$$

$\therefore$ $\triangle CGD \equiv \triangle CGE$, so that

$\angle CGD = \angle CGE$ and these angles being adjacent angles formed by one straight line standing on another straight line, are right angles.

$\therefore$ CG is perpendicular to AB.

NOTE. In practice, it is convenient to include in the construction, the directions for bisecting DE thus: With centres D, E and radii equal to DC draw arcs to cut again at F. Join CF.

Then CF is perpendicular to AB and passes through G.

EXERCISE XXI.

1. Draw any triangle and its altitudes. Do the three altitudes meet in a point? Examine the position of this point in (i) a right-angled triangle, (ii) an acute-angled triangle, (iii) an obtuse-angled triangle. (*The point of intersection of the altitudes is called the* **ortho-centre**.)

2. Perform the following construction for drawing a square on a given straight line AB :—

At B erect a perpendicular BC to AB and make BC = AB.

Bisect the angle ABC by the straight line BD.

With centre C and radius = CB, draw an arc to cut BD again at D. Join AD.

Then ABCD is the required square.

Prove the construction.

3. Draw any angle and bisect it internally as well as externally. Take some points on these bisectors and draw perpendiculars from them on the arms of the angle. Are these perpendiculars equal? Give reasons for your answer.

4. Find the image of a straight line AB in another straight line CD, by drawing perpendiculars from A and B on CD and producing them to their own lengths on the other side of CD. Prove that the image of AB thus obtained is equal to AB. Check by measurement.

5. As in Ex. 4, construct the image of a triangle ABC in a straight line XY. Show that the image is congruent to the original.

6. Draw an angle AOB. Take a point P within the angle. Through P draw a straight line QPR meeting OA, OB in Q, R such that $OQ = OR$.

Draw another straight line to meet OA in Q' and BO produced in R' such that $OQ' = OR'$.

7. Verify the following constructions for drawing the perpendicular to a given straight line AB from a given point P outside it:

(i) With centre P and a convenient radius draw an arc to cut AB at X and Y. Bisect the angle XPY. The bisector is perpendicular to AB.

(ii) With centre P and radius equal to two convenient units draw an arc to cut AB at X. On PX, take PY equal to 1 unit. With centre Y and radius YP draw an arc to cut AB at Q. Then PQ is perpendicular to AB.

(iii) Join AP. With centre A and radius AP draw an arc to cut AB at C. On AC as diameter, describe a semi-circle to cut AP at D. From AC cut off $AQ = AD$. Join PQ. PQ is perpendicular to AB.

(iv) Take any point O in AB. With O as centre and any radius, draw a circle to cut AB at C and D.

Join PC and PD and produce them, if necessary, to cut the circle again at E and F. Join ED and CF and produce them if necessary, to meet at Q.

Join PQ. PQ is perpendicular to AB.

Which of the above constructions do you prefer and why? Which of them is easier to prove?

Prove the constructions (i), (ii) and (iii).

N. B. The construction (iv) depends on the fact that *the altitudes of a triangle are concurrent.*

8. Find the image A′ of a point A in the straight line CD. Let B be any other point in the plane of A and CD. Join A′B and produce it (if necessary) to cut CD at P.

Show that the angle APB is bisected by CD internally or externally according as B lies on the side of CD opposite to A or on the same side of CD as A.

N. B. When A and B are on the same side of CD, the broken line AP, PB is the path of a ray of light proceeding from A to B after a reflection in CD.

Construction 6

At a given point in a given straight line to make an angle equal to a given angle.

Construction 7.

To draw a straight line through a given point parallel to a given straight line.

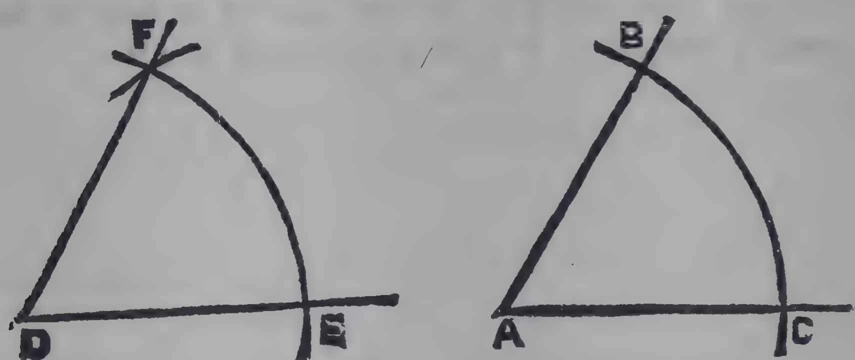

FIG. 93.

Let FDE be the given angle and AB a given straight line. It is required to make at A an angle BAC equal to the angle FDE.

Construction: With centre D and a convenient radius, draw an arc to cut DE, DF at E, F respectively. With A as centre and radius equal to DF, draw an arc BC to cut AB at B. Then, with B as centre and radius = FE, draw an arc to cut the arc BC at C. Join AC.*

Then $\angle$ BAC is the required angle.

Proof: Join EF, BC

In the $\triangle$s ABC, DFE,

AC = DE
AB = DF
BC = FE.

---

* This construction is due to Apollonius (225 B. C.) In Euclid's method, DF $\neq$ DE.

$\therefore \quad \Delta ABC \equiv \Delta DFE$, so that

$\angle BAC = \angle FDE$.

Next, let AB be the given straight line and C a point outside it. It is required to draw a straight line through C parallel to AB.

FIG. 94.

Construction: Join AC. At C in AC make $\angle ACF = \angle BAC$ on the side of AC opposite to B (by the method explained above).

Then CF is parallel to AB.

Proof: $\angle ACF = \angle BAC$ (alternate angles).

$\therefore$ CF is parallel to AB.

EXERCISE XXII.

1. Copy a given triangle ABC using one side and two angles.

2. Copy a given quadrilateral, using one side and four suitable angles.

3. Verify and prove the following constructions for drawing a parallel to a given straight line AB through a given point P :

(i) With P as centre and a convenient radius, draw an arc to cut AB at Q and R. Bisect $\angle QPR$ externally by the straight line PS. Then PS is parallel to AB.

(ii) Take any point Q in AB, so that PQ is not perpendicular to AB. Join QP and produce it to O. With centre O and radius OQ draw an arc to cut AB again at R. Join OR and from it cut off OS = OP. Join PS. Then PS is parallel to AB.

(iii) Take any point Q in AB. With centre Q and radius QP draw an arc to cut AB at R. With centre P and radius PQ draw an arc QS. Again with centre Q and radius equal to PR draw an arc to cut the arc QS at S. Join PS. Then PS || AB.

(iv) From P draw PQ perpendicular to AB. At P in PQ erect PS perpendicular to PQ. Then PS || AB.

4. AB, AC are two straight lines. Find a point P in the same plane as the lines which is at a distance of 1″ from AB and AC. Show that there are 4 such points which form the vertices of a rhombus and the diagonals of the rhombus are the internal and external bisectors of the angle BAC.

5. Through a given point A outside a given straight line BC, draw two straight lines equally inclined to BC, the angle of inclination being equal to a given angle PQR.

6. A is a given point and BC a given straight line. Join AC. At A in CA make $\angle CAQ = \angle ACB$ so that Q lies in BC. Prove that $BC = AQ + QB$ or $AQ \sim QB$. Draw figures to illustrate the three different cases.

7. At the centre O of a circle make an angle AOB equal to 72° (with the help of a protractor). Make $\angle BOC$, $\angle COD$, $\angle DOE$ at O such that $\angle BOC = \angle COD = \angle DOE = \angle AOB$ (without using the protractor).

Prove that $\angle EOA = 72°$.

If A, B, C, D, E lie on the circumference of the circle, prove that ABCDE is a regular pentagon inscribed in the circle.

Give a similar construction for inscribing within the circle regular polygons of 8, 9 and 10 sides.

8. P is a given point and QR a given straight line. Draw through P a straight line PS to meet QR at S such that $\angle PSQ - \angle PSR =$ a given angle.

§ 7. Some Inequality Theorems.†

In this section, we shall consider the inequality relations between the corresponding elements of two triangles in which two pairs of elements are equal. These two equal pairs of elements may be

either I. two pairs of angles,

II. one pair of sides and one pair of angles,

or III. two pairs of sides.

I. Let ABC, A′B′C′ be two Δs in which A = A′ and B = B′. Then we know that C = C′.

It can be easily proved that if a is greater than, equal to or less than a', then b is greater than, equal to or less than b' and c is greater than, equal to or less than c'.

II. (1) Let ABC, A′B′C′ be two Δs in which A = A′ and $a = a'$.

Then C is greater than, equal to or less than C′ according as B is less than, equal to or greater than B′.

It can be shown that if $B' > B$ and $B + B' < 180°$, then $b' > b$; but $c' > c$ if C is obtuse and $C + C' > 180°$, and $c' < c$ if C′ is acute and $C + C' < 180°$.

(2) Let ABC, A′B′C′ be two Δs in which A = A′ and $c = c'$.

Then as before C is greater than, equal to or less than C′ according as B is less than, equal to or greater than B′.

If $B < B'$, then $b < b'$ but a is greater than, equal to or less than a' according as $C + C'$ is greater than, equal to or less than 180°.

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† This section may be omitted by the beginner.

NOTE. The proofs of the results given above are left as exercises to the student.

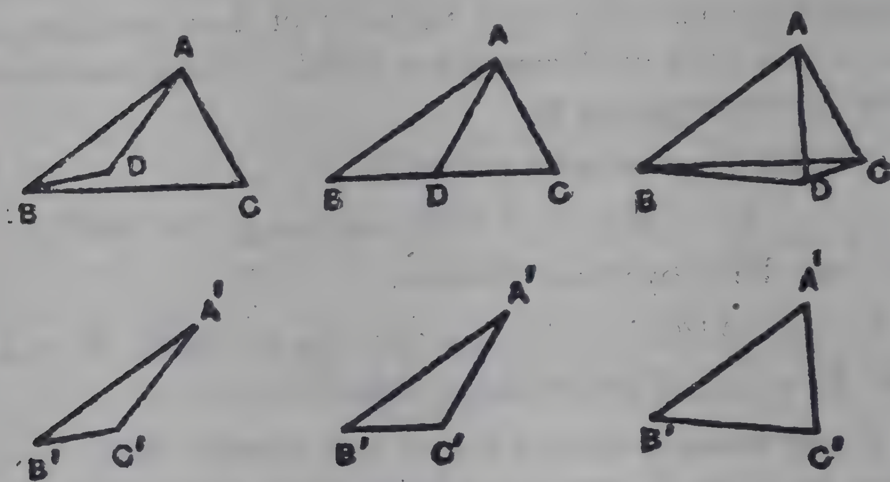


FIG. 95.

III. Let ABC, A′B′C′ be two Δs in which $b = b'$ and $c = c'$.

We shall now prove that if A > A′, then $a > a'$ and conversely, if $a > a'$, then A > A′; and establish the relations between the other pairs of elements in the two triangles.

Let $b < c$. Then $b' < c'$ and the angles B and B′ are both acute.

$\therefore \quad B + B' < 180°$.

If Δ A′B′C′ be applied to ΔABC so that A′ falls on A and A′B′ on AB and the triangle on the same side of AB as the vertex C, then B′ coincides with B and there are *three* possible positions for C′, *viz.*, (i) within Δ ABC; (ii) on BC; (iii) within $\angle$BAC but on the side of BC opposite to A. (Vide fig. 95.)

Let ABD be the new position of the ΔA′B′C′.

In case (i) since AD + DB < AC + CB and AD = AC,

$$\therefore \quad BD < BC.$$

$$\therefore \quad a > a'.$$

In case (ii), obviously $a > a'$. In case (iii), join DC.

Since AD = AC, ∠ACD = ∠ADC.

But ∠BDC > ∠ADC.

$\therefore$ ∠BDC > ∠ACD.

Hence, with greater reason, ∠BDC > ∠BCD.

$\therefore$ BC > BD,

i.e., $a > a'$.

Conversely, if $a > a'$, then A > A′.

If not, let A = A′, then $a = a'$ which is against the hypothesis. $\therefore$ A ≠ A′.

Let then A < A′, in which case $a < a'$ by the theorem just proved. But, by hypothesis $a > a'$.

$\therefore$ A is neither equal to nor less than A′.

Hence A > A′.

Cor. 1. In the cases (i) and (ii), it is evident that C < C′. We shall now show that it is also true in case (iii) and so it is always true.

Apply the △ A′B′C′ to AB so that A′ may fall on A, A′B′ on AB and C′ on the side of AB opposite to C. Then B′ coincides with B. Let ABD be the new position of the △ A′B′C′. Join DC.

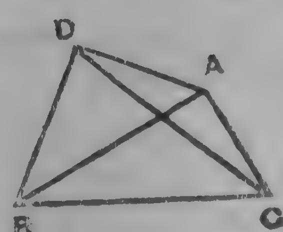


FIG. 95.

Proof: Since BC > BD

$\angle BDC > \angle BCD$

and since AC = AD

$\angle ADC = \angle ACD.$

$\therefore\ \angle BDA > \angle BCA$

i.e., $C' > C$.

$\therefore\ C < C'$.

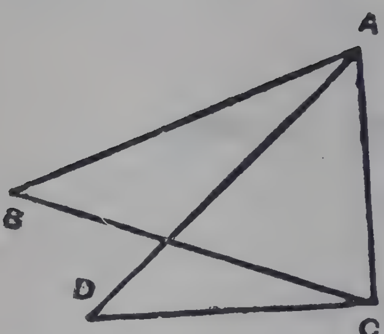


FIG. 97.

Cor. 2. Since $C < C'$, if we apply $\Delta A'B'C'$ to ΔABC so that $A'C'$ coincides with AC and B' falls on the same side of AC as B, then B' occupies the position D on the side of BC opposite to A. (Vide fig. 97.)

EXERCISE XXIII.

1. ABC, $A'B'C'$ are two triangles in which $a = a'$, $A = A'$, $B + B' < 180°$, $C + C' < 180°$. Prove that b is greater than, equal to or less than b' according as c is less than, equal to or greater than c'.

2. ABC, $A'B'C'$ are two triangles in which $a = a'$, $A = A'$ and $B + B' = 180°$. Prove that A, A' are acute angles, $C + C' < 180°$ and $b = b'$ while c may be greater or less than c'.

If $c = c'$, prove that the triangles must be right-angled and congruent.

3. ABC, $A'B'C'$ are two triangles in which $a = a'$, $A = A'$ and $B + B' > 180°$. Prove that A, A' are acute angles, and $C + C' < 180°$.

Also, prove that b is greater than, equal to or less than b' according as c is greater than, equal to or less than c'.

4. By the Law of Converses, prove that the converses of Ex. (1), (2), (3) are also true.

5. ABC, ADE are two triangles, such that $\angle BAC = \angle DAE$ and B, C, D, E are points in order on the same straight line.

Prove that AB is greater than, equal to, or less than AE according as BC is greater than, equal to, or less than DE.

Hence deduce that two triangles of equal altitudes are congruent, if their bases and vertical angles are equal.

6. ABC is a triangle in which AD is the median bisecting the base BC at D.

Prove that BC is greater than, equal to, or less than 2AD according as $\angle A$ is greater than, equal to, or less than 90° and conversely.

7. ABC, DEF are two triangles in which $AB > DE > DF$ and $AC = DF$. If $\angle BAC > \angle EDF$, prove that $BC > EF$.

8. ABC, DEF are two triangles right-angled at C and F respectively. If $AB = DE$ and $BC < EF$, prove that $AC > DF$ and $\angle ABC > \angle DEF$.

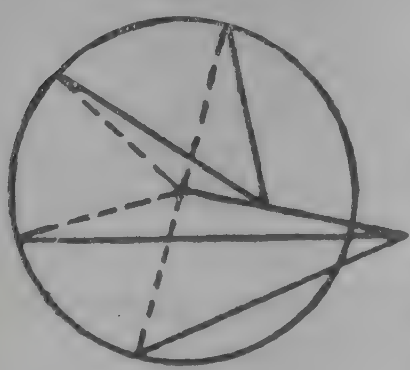


FIG. 98.

9. If from any point within or outside a circle, two unequal straight lines be drawn to the circumference, prove that the greater is that which subtends the greater angle at the centre and conversely.

10. ABC is a triangle in which $AB > AC$. D is the middle point of BC. If P be any point in DA or DA produced, prove that $PB > PC$. Is the result true when P lies in AD produced? Give reasons for your answer.

11. Show that a side of a regular hexagon is always less than a side of a regular pentagon inscribed in the same circle.

12. ABC, DEF are two triangles in which $AB = DE$ and $BC = EF$ and $\angle ABC > \angle DEF$. If X, Y are the middle points of AC, DF respectively, prove that $BX < EY$.

13. In figure 95, prove that $BC > BD$ by bisecting $\angle DAC$ by a straight line AE meeting BC in E and joining DE.

CHAPTER VII.

PARALLELOGRAMS.

§ 1. Definitions.

Def. A quadrilateral whose opposite sides are parallel is called a parallelogram (▭*m*).

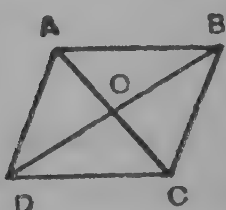


FIG. 99.

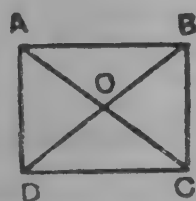


FIG. 100.

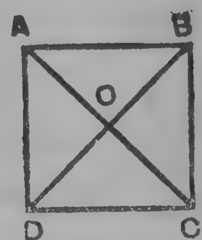


FIG. 101.

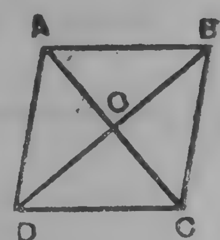


FIG. 102.

The existence of a parallelogram is easily proved by taking two straight lines AC, BD bisecting each other, say at O. Then the quadrilateral of which AC, BD are the diagonals is a ▭m.

A ▭m is sometimes referred to by the letters denoting a pair of opposite vertices. Thus the ▭m ABCD may be briefly called ▭m AC or BD.

Def. A rectangle is a parallelogram which has one angle a right angle. (Fig. 100.)

But, it can be proved that if in a parallelogram, one angle is right, the others are also right.

A rectangle is obtained by taking as diagonals two *equal* straight lines which bisect each other.

Def. A square is a rectangle in which all the sides are equal. (Fig. 101.)

Take two *equal* straight lines bisecting each other *at right angles.* Their extremities form a square.

Def. *A rhombus is a quadrilateral in which all the sides are equal.* (Fig. 102.)

Thus, a square may also be considered as a rhombus, but a rhombus is not a square.

To construct a rhombus, construct a quadrilateral with two *unequal* straight lines as diagonals *bisecting each other at right angles.*

FIG. 103.

Def. *A trapezium is a quadrilateral which has only one pair of sides parallel*

§ 2. Properties of a Parallelogram.

Theorem 17.

(i) *The opposite sides and angles of a parallelogram are equal*; (ii) *each diagonal bisects the parallelogram; and* (iii) *the diagonals bisect each other.*

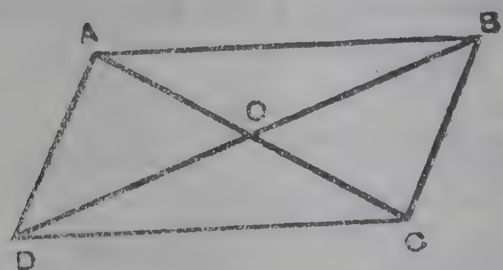


FIG. 104.

Let ABCD be a ▭m of which AC, BD are the diagonals intersecting at O.

It is required to prove:

(i) AB = CD and BC = DA;

(ii) Each of the diagonals AC, BD bisects the ▭m;

(iii) OA = OC and OB = OD.

Proof:

Since AB and CD are parallel and AC meets them,

∴ $\angle$BAC = $\angle$DCA (alt. $\angle$s).

Again, since BC and AD are parallel and AC meets them,

∴ $\angle$BCA = $\angle$DAC (alt. $\angle$s).

Hence, in the Δs ABC, CDA

$$\begin{cases} \angle BAC = \angle DCA \\ \angle BCA = \angle DAC \\ AC \text{ is common} \end{cases}$$

$\therefore \quad \Delta ABC \equiv \Delta CDA$, so that

$AB = CD$, $BC = DA$, $\angle ABC = \angle CDA$ and AC bisects the ▭m.

Similarly, it can be shown that BD also bisects the ▭m and $\angle BAD = \angle DCB$.

Again in the Δs AOB, COD

$$\begin{cases} \angle BAO = \angle DCO & \text{(proved)} \\ \angle AOB = \angle COD & \text{(vert. opp. } \angle\text{s)} \\ \quad AB = CD & \text{(proved)} \end{cases}$$

$\therefore \quad \Delta\ AOB \equiv \Delta\ COD$, so that

$OA = OC$ and $OB = OD$,

i. e., the diagonals bisect each other.

Cor. 1. If one angle of a ▭m is a right angle, then all its angles are right angles and the diagonals of such a ▭m are equal.

Cor. 2. In every ▭m not containing a right angle, there are always two acute and two obtuse angles and the diagonals are unequal.

Conversely, if the diagonals of a ▭m are unequal, the parallelogram has two acute and two obtuse angles.

Cor. 3. A ▭m which has two adjacent sides equal has all its sides equal and is therefore a square or a rhombus, according as its diagonals are equal or unequal.

In this case, the diagonals divide the ▭m into four congruent △s and themselves cut at right angles; also, each diagonal bisects the angles which they join.

Example 1. *The straight lines joining two equal and parallel straight lines towards the same parts are equal and parallel.*

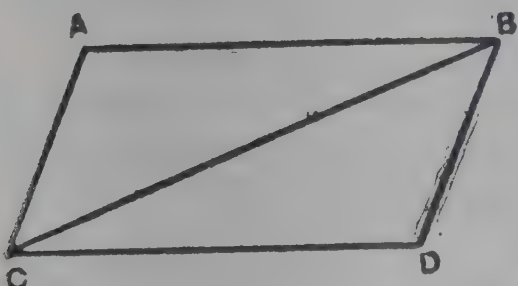


FIG. 105.

Let AB, CD be two equal and parallel straight lines joined towards the same parts by the straight lines AC, BD.

It is required to prove that AC and BD are equal and parallel.

Construction: Join BC.

Proof: In the △s ABC, DCB,

$$\begin{cases} AB = DC. \\ BC \text{ is common.} \\ \angle ABC = \angle DCB \text{ (alt. } \angle\text{s made by the transversal } BC \text{ with the parallels } AB, CD) \end{cases}$$

$\therefore \quad \triangle ABC \equiv \triangle DCB.$

so that $\quad AC = BD$ and $\angle BCA = \angle CBD$

These angles are alternate angles. Hence AC and BD are parallel.

NOTE. This proposition, according to Proclus, is the connecting link between the theory of parallels and that of parallelograms and gives the construction or origin of parallelograms.

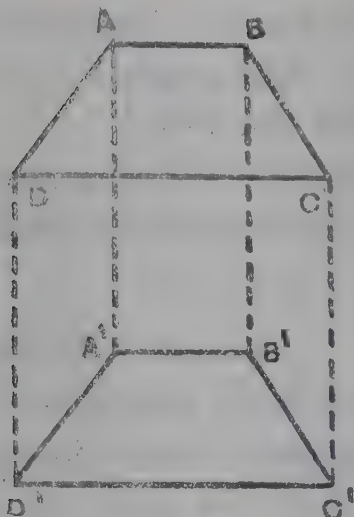


FIG. 106.

It may be used to construct congruent figures thus:—

To construct a figure congruent to ABCD, draw equal parallels AA′, BB′, CC′, DD′, in the same direction. Then the figure A′B′C′D′ ≡ the figure ABCD.

N. B. This construction is equivalent to the motion of translation of the figure ABCD in the direction AA′.

Example 2. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

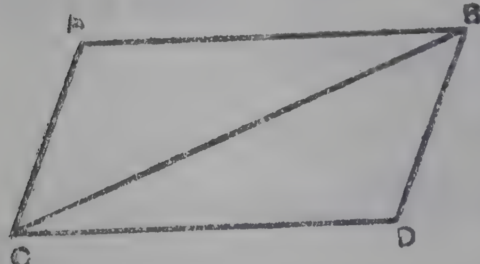


FIG. 107.

Let ABDC be a quadrilateral in which AB = CD and AC = DB.

It is required to prove that the figure is a parallelogram.

Construction: Join BC.

Proof: In the Δs ABC, DCB

$$\begin{cases} AB = CD \\ AC = DB \\ BC \text{ is common.} \end{cases}$$

∴ Δ ABC ≡ Δ DCB, so that

(i) ∠ABC = ∠DCB, which are alt. ∠s.

∴ AB and CD are parallel.

and (ii) ∠ACB = ∠DBC. These are also alt. ∠s

and ∴ AC and DB are parallel.

∴ The figure ABDC is a ▭m.

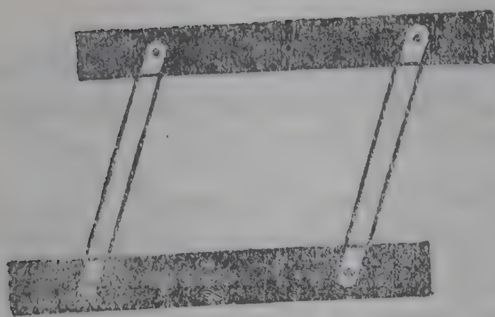


FIG. 108.

NOTE. This proposition gives a simple test for a parallelogram.

The common folding parallel ruler (Fig. 108) involves this principle.

EXERCISE XXIV.

1. Prove that parallel st. lines are everywhere equi-distant.

2. ABC is a triangle. DE is drawn parallel to BC to cut AB in D and AC in E. Prove that DE < BC.

3. ABC is a triangle. D, E, F are points in the sides BC, CA, AB respectively such that the sides of the triangle DEF are parallel to those of the triangle ABC.

Prove that D, E, F are the mid. points of the sides.

4. If a quadrilateral has three angles right, show that it is a rectangle.

5 If a parallelogram has two adjacent sides and angles equal, show that it is a square.

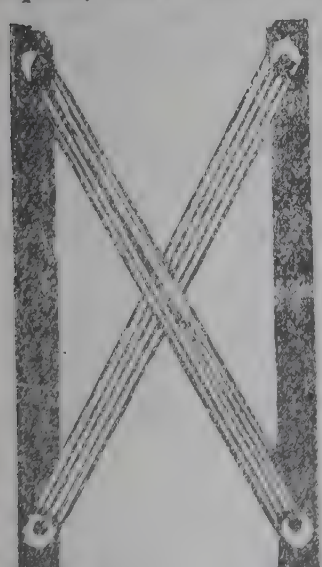


FIG. 109.

6 If in a quadrilateral the diagonals bisect each other, show that it is a parallelogram.

NOTE. Sometimes, a parallel ruler is constructed on this principle. But unless the workmanship is very good, the parallel ruler of the form shown in the figure will not be accurate. Folding iron-gates also illustrate the same principle.

7. Explain the principle underlying the following method of finding not only the distance but the direction between two inaccessible points X, Y.

Walk along a straight line AB and fix pts P, Q in it such that (PX, PB), (QY, QB) may be seen along two sides of a rectangle. Find R the mid. pt. of PQ. Walk in the directions YR, XR produced to X′, Y′ such that X′ is in a line with XP produced and Y′ is in a line with YQ produced.

Then X′Y′ is equal and parallel to XY.

8. Show that a circle can be drawn to pass through the vertices of an equiangular parallelogram.

9. ABCD is a parallelogram. O is any point on the diagonal BD. Draw POQ, ROS parl. to AD, AB respectively so that P, Q may lie on AB, CD and R, S in AD, BC. Prove that ▭m APOR = ▭m OSCQ.

10. If in a convex quadrilateral, one pair of opposite sides be equal and also one pair of opposite angles, is the quadrilateral necessarily a parallelogram? Examine the cases where (1) the equal sides are not greater than the other sides of the quadrilateral and (2) the equal angles are not acute.

11. Construct a parallelogram ABCD witth the following data:

(i) AB = 2″, AC = 3″, AD = 4″

(ii) AC = 4″, BD = 3·5″, AB = 3″.

(iii) ∠BAC = 30°, BD = 10 cms., AC = 12 cms.

(iv) AC = 8 cms, BD = 9 cms., the angle between AC and BD = 90°.

(Protractors should not be used.)

12. Use the properties of a parallelogram to bisect a given st. line.

13. Prove that every st. line through the centre (*i.e.*, the point of intersection of the diagonals) of a parallelogram bisects the parallelogram.

14. Through a point P within an angle AOB, draw a st. line QPR to meet OA in Q and OB in R such that $PQ = PR$.

15. Inscribe a rhombus within a triangle such that two of its sides may lie on the sides of the triangle.

16. ABCDEF is a hexagon in which opposite sides are parallel. Show that the opposite angles of the hexagon are equal. If one pair of opposite sides be equal, show that the remaining pairs of opposite sides are also equal and the diagonals joining pairs of opposite vertices bisect each other. Show also that the remaining six diagonals form two triangles with corresponding pairs of sides equal and parallel.

§ 3. Theorem 18. (The Intercept Theorem).

If the intercepts made by three or more parallel straight lines on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

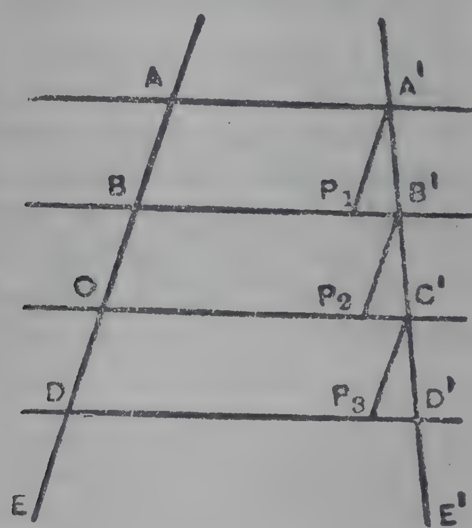


FIG. 110.

Let AA′, BB′, CC′, DD′,...be three or more parallel st. lines making equal intercepts AB, BC, CD, ... on any transversal AE and A′B′, B′C′, C′D′, ... the intercepts on any other transversal A′E′. It is required to show that A′B′ = B′C′ = C′D′ = ...

Construction: Through A′, B′, C′,...draw parallels $A'P_1$, $B'P_2$, $C'P_3$, ... to AE to meet BB′, CC′, DD′,... at P_1, P_2, P_3, ... respectively.

Proof: Since the figure $A'P_1BA$ is a parallelogram, the opposite sides are equal.

$\therefore \quad A'P_1 = AB.$

Similarly, $B'P_2 = BC$, $C'P_3 = CD$ and so on

Since $AB = BC = CD = \ldots\ldots$

it follows that $A'P_1 = B'P_2 = C'P_3 = \ldots\ldots$

Again, since $A'P_1$, $B'P_2$, $C'P_3$, ... are all parallel to AE, they are parallel to one another; and since the transversal $A'E'$ cuts them, the corresponding angles are equal

$$\therefore \quad \angle P_1A'B' = \angle P_2B'C' = \angle P_3C'D' = \ldots\ldots$$

For a similar reason

$$\angle P_1B'A' = \angle P_2C'B' \equiv \angle P_3D'C' = \ldots\ldots$$

Hence in the $\triangle$s $P_1A'B'$, $P_2B'C'$, $P_3C'D'$,......

$$\begin{cases} A'P_1 = B'P_2 = C'P_3 = \ldots\ldots \\ \angle P_1A'B' = \angle P_2B'C' = \angle P_3C'D' = \ldots\ldots \\ \angle P_1B'A' = \angle P_2C'B' = P_3D'C' = \ldots\ldots \end{cases}$$

$$\therefore \quad \triangle P_1A'B' \equiv \triangle P_2B'C' \equiv \triangle P_3C'D' \equiv \ldots\ldots$$

so that $\qquad A'B' \equiv B'C' = C'D' = \ldots\ldots$

and $\qquad P_1B' = P_2C' = P_3D' = \ldots\ldots$

Cor. 1. **A system of equidistant parallel st. lines makes equal intercepts on any transversal which cuts it.**

Cor. 2. **If through the mid. pt. of a side of a triangle a st. line be drawn parallel to the base, it bisects the third side and is equal to half of the base.**

Cor. 3. **In a trapezium, the st. line through the mid. pt. of one of the oblique sides parallel to the parallel sides bisects the other oblique side.**

Cor. 4. In the figure of Th. 18, since $P_1B' = BB' - AA'$, $P_2C' = CC' - BB'$, etc., and $P_1B' = P_2C' = \ldots$, it follows that $BB' - AA' = CC' - BB' = DD' - CC' = \ldots$

$\therefore \quad AA' + CC' = 2BB'; \quad BB' + DD' = 2CC'$ and so on.

NOTE 1. The above property of equidistant parallels is useful to divide a given st. line AB into any number of equal parts (say 5). Take a piece of ruled tracing paper and place it on the given line so that one of the parallels passes through A and the 6th parallel from it passes through B. The points where the intermediate 4 parallels meet AB are the required points of division. (*Vide* fig. 111).

NOTE 2. About 900 A.D, the Arab Al-Nairizi gave an elegant construction to divide a given st. line into any number of equal parts, which requires only one measurement repeated and the properties of parallels proved in Th. 18. (*Vide* Construction 8.)

Construction 8.

Tc divide a given straight line into any number of equal parts (say 5).

First method: (Al-Nairizi's method).

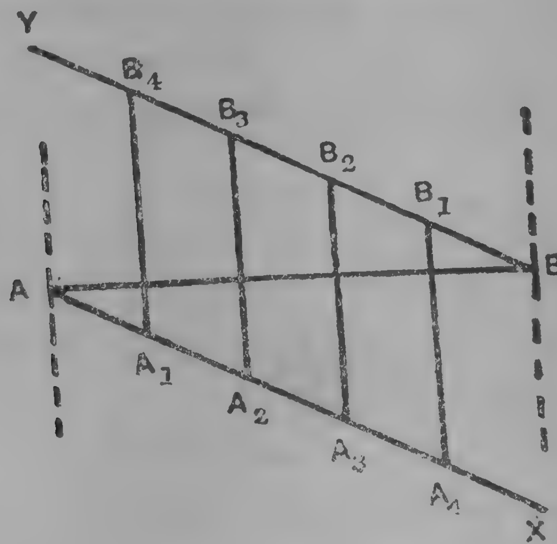


FIG. 111.

Let AB be the given st. line to be divided into 5 equal parts.

Construction. Through A, B, draw two convenient parallels AX, BY and on them set off equal lengths $\{AA_1, A_1A_2, A_2A_3, A_3A_4\}$, $\{BB_1, B_1B_2, B_2B_3, B_3B_4\}$ respectively.

Join A_1B_4, A_2B_3, A_3B_2, A_4B_1.

Then these lines divide AB into 5 equal parts.

Proof. Since A_1A_2 is equal and parallel to B_4B_3, A_1B_4 and A_2B_3 are parallel.

Similarly, A_2B_3, A_3B_2, and A_4B_1 are all parallel to one another.

If we suppose parallels to A_1B_4 drawn thro' A and B also, we have, by the **intercept theorem**, equal intercepts made by these parallels on the transversal AB.

Second method :

Construction. Draw any straight line AX making any convenient angle with AB and on it set off 5 equal lengths AA_1, A_1A_2, A_2A_3, A_3A_4, and A_4A_5. Join BA_5 and through A_1, A_2, A_3, A_4 draw parallels to BA_5 to cut AB.

Then these 4 parallels divide AB into 5 equal parts.

Proof. If, as before, we suppose a parallel drawn thro' A to BA_5, we easily see by the **intercept theorem**, that AB is divided into 5 equal parts, by the 6 parallels which make equal intercepts on the transversal AX.

EXERCISE XXV.

1. Draw a straight line AB and divide it into 7 equal parts.

2. (i) Cut off from a given straight line AB, two-fifths of it.

(ii) Draw a straight line CD and divide it in the ratio 2 : 3 : 5.

3. ABC is a triangle. A_1B_1, A_2B_2, A_3B_3, A_4B_4 are drawn parallel to BC meeting AB and AC at (A_1, A_2, A_3, A_4) and (B_1, B_2, B_3, B_4) respectively.

If $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4B$, prove that $A_1B_1 = 1/5$ BC, $A_2B_2 = 2/5$ BC, $A_3B_3 = 3/5$ BC and $A_4B_4 = 4/5$ BC.

4. ABCD is a trapezium in which AB and CD are parallel. EF is a straight line parallel to AB cutting AD produced in E and BC produced in F. Given AB = 5 cms., BC = 4 cms., CD = 7 cms., AD = 3 cms., and DE = 6 cms., construct the trapezium and calculate CF and EF.

5. ABC, DEF are two triangles in which $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$. If $AB = 3DE$, prove that $BC = 3EF$ and $AC = 3DF$.

6. Construct a quadrilateral ABCD, given $AB = 8$ cms. $BC = CD = 10$ cms., $AC = 9$ cms., $AD = 6$ cms.

Construct another quadrilateral of half the linear dimensions, similar to ABCD.

7. Prove that the straight line joining the mid. pts. of two sides of a triangle is parallel to and half of the third side.

(Let D, E be the mid. pts. of the sides AB, AC of a ΔABC. Join DE and produce it to F making $EF = ED$. Join FC. Prove that FC is equal and parallel to AD and therefore to DB, so that DBCF is a $\square^m$ in which the opp. sides DF and BC are equal and parallel.)

8. Prove Ex. 7. by the method of *Reductio ad absurdum* using the **intercept theorem.**

9. ABCD is a trapezium in which AB and CD are parallel. E, F, G, H are the mid. pts. of AD, BC, CA, BD respectively. Prove that E, F, G, H are collinear and EF is parallel to AB or CD. Show also that $EF = \frac{1}{2}(AB + CD)$; $EH = GF$; and $GH = \frac{1}{2}(AB \sim CD)$. If K, L be the mid pts. of AB, CD, prove that the figures EKFL and HKGL are $\square$ms and KL bisects both EF and GH.

Investigate the corresponding results when ABCD is an ordinary quadrilateral, i.e. AB is not parallel to CD, and state which of the above results remain unaltered and which have to be modified.

10. The figure below is called *a diagonal scale* which shows inches, tenths and hundredths of an inch.

There are 11 equidistant horizontal parallel lines, the distance between the top and the bottom lines being $1''$. The top and bottom lines are divided into inches and the corresponding points of division joined by lines perpendicular to the horizontal lines. The inch-division on the

extreme left of the bottom line is sub-divided into tenths of an inch marked 0, 1, 2, 3, 4, ... from right to left and

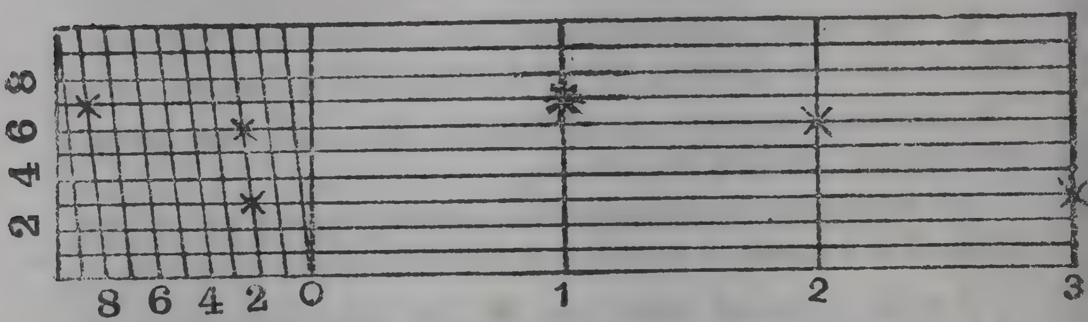


FIG. 112.

the inch-divisions to the right of 0 are marked 1, 2, 3, ... The extreme left inch-division on the top line is also sub-divided into tenths of an inch and the points of division joined diagonally to the points of division in the bottom line as shown in the figure. The figure thus obtained is a diagonal scale.

Show that the diagonal-line through '0' recedes from the perpendicular through '0' by horizontal distances which increase successively by 1/100″ as we cross each horizontal line from the bottom-line to the top-line.

What is the distance between the pairs of star-marks on each of the three horizontal lines in the figure?

With the help of the diagonal scale, draw straight lines of the following lengths:

2·56″, ·73″, 3·48″, 1·21″, ·09″.

Find the number of inches, correct to two places of decimals, in the following lengths:

5 cms., 6·7 cms., 9·3 cms, 12·6 cms.

11. A map is constructed on a scale of 5 cms. to the mile. Construct a scale for reading off distances on the map which correspond to as small a length as 20 yards.

12. Construct a diagonal scale of yards, feet, and inches.

CHAPTER VIII.

CONSTRUCTION OF TRIANGLES AND QUADRILATERALS FROM GIVEN DATA.

§ 1. General Remarks.

1. In geometrical constructions, the only instruments that we are allowed to use are:

(i) the ruler, for joining points and producing straight lines and

(ii) the compasses to draw circles with a given centre and radius. These instruments should not be used for any other purpose.

The scale and the protractor should not be used unless they are found absolutely necessary as when numerical data are given.

2. In discovering the clue to the construction of a figure from given data, it is advisable to draw a rough figure supposed to satisfy the given conditions, and mark therein the given data, so that we may be able to perceive what portions of the figure can be first constructed directly from some of the given data, and how the figure can be subsequently completed. The rough figure is absolutely necessary to get a start in the construction as it helps us to see all the given data together in their proper perspective. Sometimes, the given data cannot be directly marked in the figure but may suggest simple constructions to be made with respect to the hypothetical figure, which will enable us to visualise the geometrical relations between the data

and the several parts of the required figure. Thus, a study of the internal structure of the figure will reveal the course one should take in solving the problem.

The method outlined above is known as the method of **analysis**, wherein we assume the result sought to be already there and see what results follow. Then we reverse the process and verify whether it leads back to the required result. If it does, we explain the process and give the proof. This is the method of **synthesis**, which suppresses the clue by which we are able to get a start in our reasoning.

3. In the following sections, we give the method of analysis, the method by which the steps in the required construction are suggested, and subsequently present the construction in the synthetic form, the form which builds up known results into the required structure. The proofs of the construction are left to be supplied by the pupil.

4. Sometimes, the constructions may lead to more than one figure satisfying the data and all such figures are not necessarily congruent. In such cases, the problem is said to admit of *more than one* solution.

Again, it may become impossible, in certain cases, to carry out the construction, as when the arcs and the straight lines (by means of which the construction is effected) do not meet. The construction is then said to *fail*. But when the construction fails, the problem itself or the solution need not fail; for it may be possible to adopt a different construction which leads to the successful solution. But if the solution fails, naturally, all constructions intended to lead to the solution must also fail.

The failure of a solution is due to some geometrical discrepancy among the data; for example, it is impossible to construct a Δ with sides 1, 2, and 4 units; while the failure of a construction may be due merely to the unsuitability of the method of construction for the given set of data.

5. If the data are insufficient, as when we are asked to construct a Δ with two angles, we get an infinite number of solutions. Sometimes, even when the data appear to be sufficient, we get an infinite number of solutions. This is due to the data being in a peculiar situation. Again, on account of some special relations among the data, the number of solutions may vary and the forms of the solutions may also be different.

6. We shall illustrate * the remarks in paras 4 and 5 (above) with the following problem.

Problem: *To draw a straight line through a given point P, such that the perpendiculars on it from two given points A, B may be equal.*

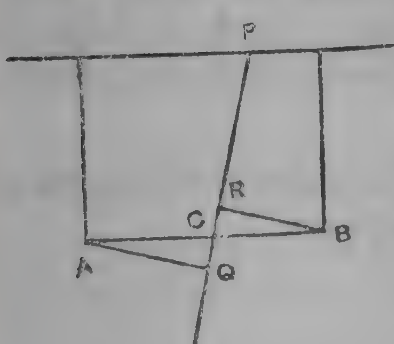


FIG. 113.

Analysis: Suppose QR is the required straight line through P and AQ, BR perpendicular to it.

Now, there are two cases to consider:

(i) the points A, B may be on the same side of QR

(ii) the points A, B may be on opposite sides of QR.

* For further illustrations, *vide* § 2, p. 166.

In case (i), it is easily seen that AB is parallel to QR and in case (ii), QR bisects AB *i.e.* passes through the middle point of AB. Here we have the clue to the required construction.

Construction: Bisect AB at C. Join PC.

Through P draw a straight line parallel to AB.

Then this straight line and PC are the required straight lines.

Remarks: 1. There are *two* solutions to this problem.

2. The number of solutions reduces to *one*, when P, A, B are collinear. In this case AB itself is the required line.

3. The number of solutions becomes *infinite*, if P is the middle point of AB. For then, every straight line through P satisfies the required condition, *viz.* the perpendiculars on it from A and B are equal.

§ 2. Construction of Triangles.

1. We shall adopt in this section the usual notation for the elements of a triangle ABC; *viz.* A, B, C for the angles and a, b, c for the sides opposite to A, B, C respectively.

2. The four fundamental cases in the construction of triangles with three given elements are:

Construction 9.

(i) *To construct a triangle, given b, c, A (two sides and the included angle)*;

(ii) *To construct a triangle, given B, C, a or b (two angles and one side)*;

(iii) *To construct a triangle, given a, b, c (three sides)*;

(iv) *To construct a triangle, given b, c, B (two sides and an angle opposite to one of them).*

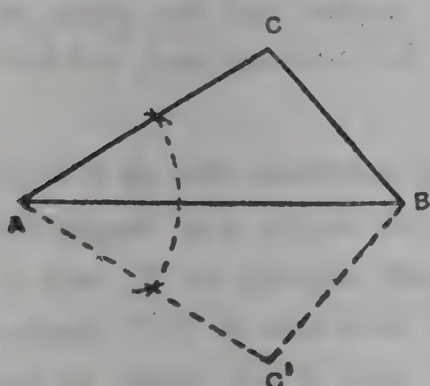


FIG. 114.

(i) Draw AB = c and at A in AB make $\angle$BAC equal to the given angle (either by means of a protractor if the measure of the angle is known or by Construction 6.) Make AC = b. Join BC. Then ABC is the Δ required. (As the angle BAC can be made on either side of AB, there are apparently two solutions but practically only *one*, the Δs obtained being congruent)

(ii) Draw BC = a; at B and C on the same side of BC make angles DBC, ECB equal to the given angles. Produce, if necessary, BD and CE to meet at A. Then ABC is the Δ required [fig. 115. (1)].

If BD, CE are either parallel [fig. 115. (2)] or do not meet on the side on which the angles are drawn [fig. 115. (3)], the solution fails and the triangle with the given data is impossible. This will happen if B + C $\nless$ 180°.

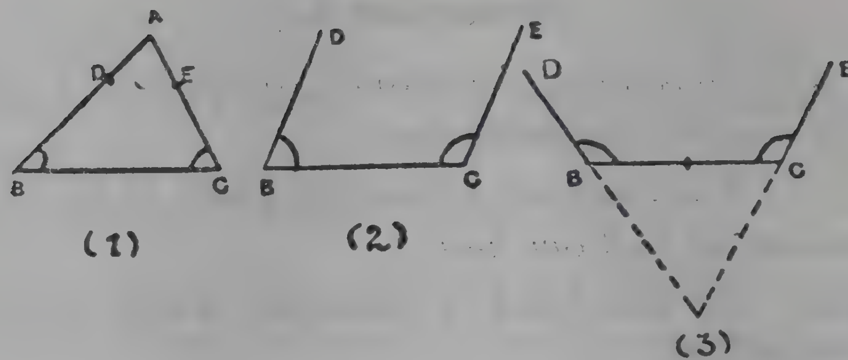


FIG. 115.

In the case where the solution is possible, there are apparently two solutions, as the angles can be made on either side of BC; but since the Δs are congruent, in effect there is only one solution.

If side b be given instead of a, construct the Δ B′CA′ as before, taking any length for B′C and making at B′ and C on the same side of B′C angles A′B′C and A′CB′ equal to the given angles. On CA′ or CA′ produced, take CA $= b$ and through A draw AB parallel to A′B′ to meet CB′ or CB′ produced at B.

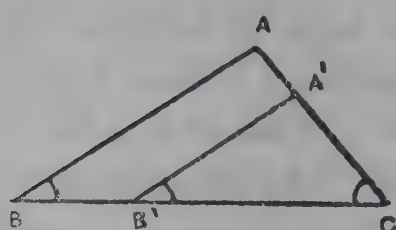


FIG. 116.

Since $\angle$ABC $=$ $\angle$A′B′C (corresponding angles), and AC $= b$, ABC is the required Δ.

(iii) Draw BC $= a$. With B and C as centres and radii equal to c and b respectively, draw arcs to cut at A, A′. Join AB, AC, A′B, A′C [fig. 117. (1)].

ABC, A′BC are the two required Δs and since they are congruent, there is in effect only one solution.

If AB + AC = BC, the two arcs meet in BC itself

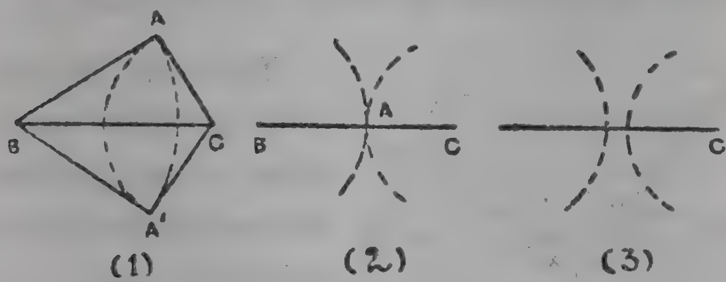


FIG. 117.

[fig. 117. (2)] and there is no proper triangle formed. This may be called the degenerate case.

If AB ~ AC > BC or AB + AC < BC, the two arcs fail to cut and both the solution and the construction fail [fig. 117. (3)].

For a successful solution, therefore, we should have

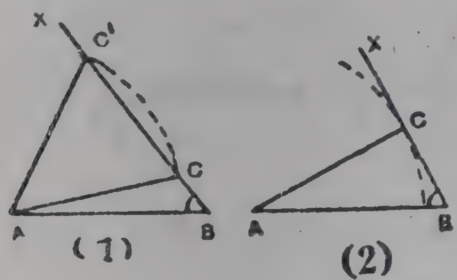
$$b \sim c < a < b + c.$$

(iv) For this case, it is advantageous to draw a rough figure and mark therein the data. The data fixed in their proper places at once suggest that we should start the construction with c and B.

Draw AB = c. At B in AB make $\angle$ABX = the given angle. With centre A and radius equal to b draw an arc to cut BX at C. Join AC. Then ABC is the required Δ.

NOTE. Six possible cases can happen as shown in the figures (1) — (6).

Case (i) The arc with centre A may cut BX in two points on the same side of AB as the angle ABX.



This will happen when $\angle B$ is acute and b less than c and greater than the perpendicular from A to BX.

There are two solutions.

FIG. 118.

Case (ii) The arc with centre A may meet BX just in one point only on the same side of AB as the angle ABX and does not meet BX again.

This happens when $\angle B$ is acute and b is equal to the perpendicular from A to BX.

There is only one solution and the triangle is right-angled.

Case (iii). The arc with centre A may cut XB in two points on opposite sides of AB or one of the points may coincide with B. This will happen when b is greater than

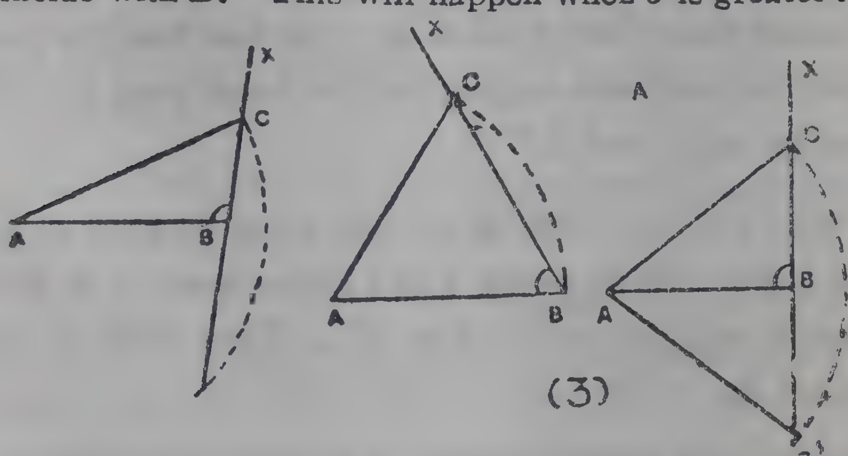


FIG. 118.

or equal to c. In this case also, there is only one solution, the solution on the other side of AB not being admissible, since the other angle at B is different. Even if the angle at B on the other side of AB be equal to the given angle, which will be the case when $B = 90°$, the triangle on the other side of AB becomes congruent to the Δ ABC and hence is no additional solution.

Case (iv). The arc with centre A may not cut BX at all. This will happen when b is less than the perpendicular from A to BX. This is the case of no solution or a failure of solution.

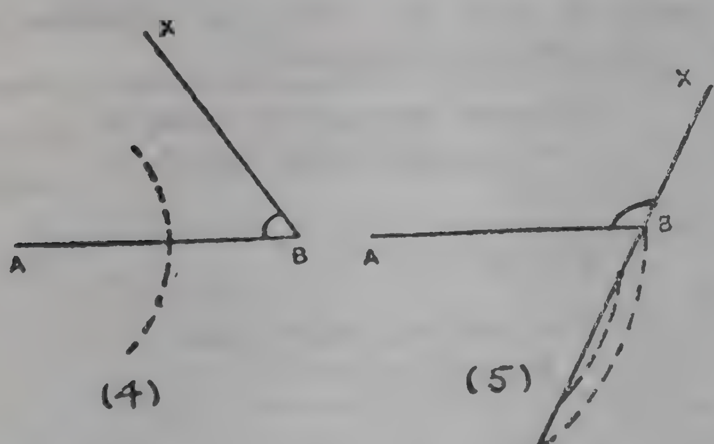


FIG. 118.

Case (v). The arc with centre A may cut XB produced at two points, both being on the side of AB opposite to the angle ABX or one of them may fall on B. This will happen when $\angle B$ is obtuse and c is greater than or equal to b. Here also, the solutions obtained are in-admissible. This is also a case of failure.

Case (vi). The arc with centre A may meet XB or XB produced in only one point and this point may be either B

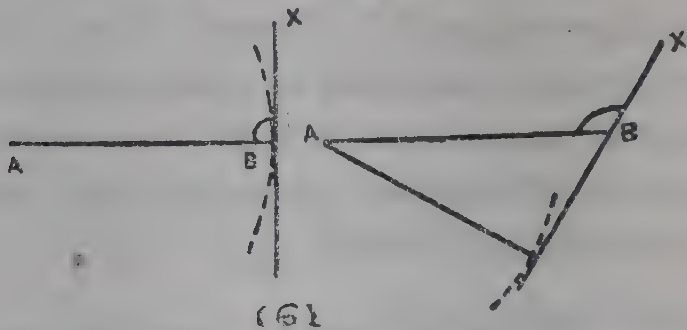


FIG. 118.

or some point in XB produced. This will happen when b is equal to the perpendicular from A to BX and $\angle B$ is either right or obtuse. This is also a case of no solution.

The cases of *zero*, *one*, and *two* solutions may be summed up in a tabular form thus:

$\angle B$	Relation between b, c and p the perpendicular from A to BX.	Number of solutions.
Acute	$b < p$ ...	0
	$b = p$ or b is greater than or equal to c ...	1
	$c > b > p$ ...	2
Obtuse	b less than or equal to p; or c greater than or equal to $b > p$ ...	0
	$b > c$ ...	1
Right	b is less than or equal to p ...	0
	$b > c$ (here $p = c$) ...	1

This exhausts all the possible cases and the Law of Converses operates.

Remarks: We gather from the above constructions that it is possible to construct a triangle, given three suitable elements. This shows that any *three magnitude conditions* properly chosen are *necessary and sufficient** to construct a triangle, because three conditions imply three relations between three suitable elements, say, the sides of the triangle, from which these latter can be derived at least theoretically from algebraic considerations.

3. Some instances of construction of triangles from three data other than three elements are given below.

---

* The propositions in Mathematics are generally of the form :

If A = B, then C = D. (i)

Here, the condition A = B is said to be ***sufficient*** for C = D for the former alone *suffices* for inferring the latter ; and the condition C = D is said to be ***necessary*** for A = B, for the former ***necessarily*** follows from and is implied in the latter.

If the converse of (i) be also true, ***viz.***, if C = D, then A = B then each is a necessary and sufficient condition for the other. To take a non-mathematical illustration, since the proposition "If A is the father of B, then B is the son of A," and its converse are both true, it follows that, A being the father of B is the necessary and sufficient condition for B being the son of A and ***vice versa***.

In every theorem, the data give generally the sufficient conditions for the conclusion to be true, while the conclusion may be considered to give the necessary conditions for the data to be consistent.

Construction 10.

To construct a triangle, given two angles and (i) *the sum,* (ii) *the difference of two sides,* (iii) *the perimeter,* (iv) *the difference between the sum of two sides and the third side.*

(i) Let B, C and $b + c$ be given.

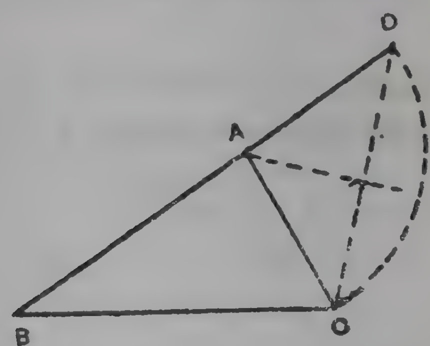


FIG. 119.

Analysis : Suppose the triangle ABC solved. Produce BA to D making AD = AC. Join DC. Then in the $\triangle$ BDC, the three elements BD, $\angle BDC = 90° - \frac{B + C}{2}$, $\angle DBC$ are known. Hence the $\triangle$ BDC can be constructed. Since DA = AC the perpendicular bisector of DC passes through A.

Construction: Draw $BD = b + c$ and at B and D in BD make angles equal to $\angle B$ and $90° - \frac{B + C}{2}$ respectively to form the $\triangle$ BCD. Draw the perpendicular bisector of CD to cut BD at A. Join AC.

ABC is the required $\triangle$.

(ii) Let B, C and $b - c$ be given.

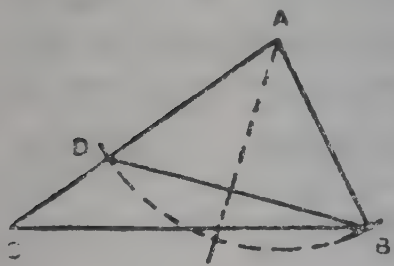


FIG. 120.

Analysis: Suppose the Δ ABC solved. Let AC > AB. From AC cut off AD = AB. Join BD. Then in the ΔCDB the three elements CD, $\angle$C, $\angle$CBD$=(B-C)/2$ are known. Hence the Δ can be constructed and since AD = AB, the perpendicular bisector of BD fixes the position of A.

(iii) Let B, C and $a + b + c$ be given.

Analysis: Suppose the Δ ABC constructed. Produce

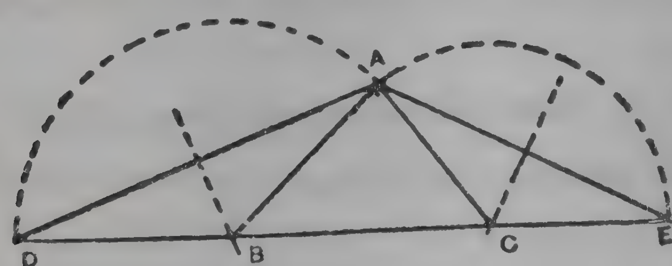


FIG. 121.

BC both ways to D and E so that BD = BA and CE = CA (as in the fig.) Join AD, AE.

Then $\angle ADB = \frac{B}{2}$ and $\angle AEC = \frac{C}{2}$ and

$DE = a + b + c$.

$\therefore$ The Δ ADE can be constructed.

The perpendicular bisectors of AD, AE meet DE in B, C. Hence, the positions of B, C can be fixed in DE.

(iv) Let B, C, $b + c - a$ be given.

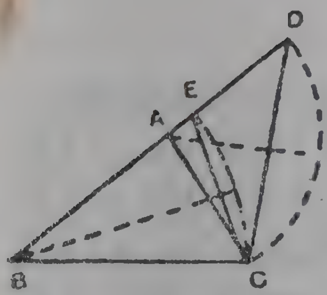


FIG. 122.

Analysis: Suppose the $\triangle$ ABC known. Produce BA to D making AD = AC and from BD cut off BE = BC. Join DC, EC. Then ED = $b + c - a$ is known. $\angle$EDC = $\frac{1}{2}\angle BAC = 90^\circ - \frac{B+C}{2}$.

Since BE = BC, $\angle BEC = 90^\circ - \frac{1}{2}B$,

therefore $\angle DEC = 90^\circ + \frac{1}{2}B$.

Thus in the $\triangle$ DEC, one side and two angles are known and therefore the $\triangle$ can be constructed.

Since DA = AC and BE = BC, the perpendicular bisectors of DC and EC meet DE or DE produced in A and B.

§ 3. Some Miscellaneous Constructions.

(1) *To construct a triangle given* $b + c$ *or* $b \sim c$, $B \sim C$, a.

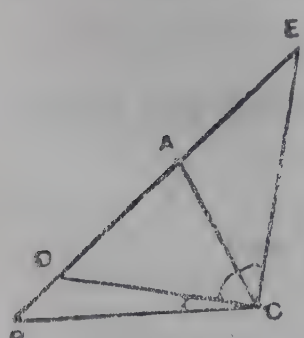


FIG. 123.

Analysis: Let ABC be the required $\triangle$. From AB cut off AD = AC and in BA produced take AE = AC. Join CD, CE. Then it is easily seen that BD = $b \sim c$, BE = $b + c$, $\angle BCD = \frac{1}{2}(B \sim C)$ and $\angle DCE = 90^\circ$.

Since BC is also given, the data are sufficient to determine either the $\triangle$ BCD or the $\triangle$ BCE. Then the perpendicular bisector of CD or CE fixes the position of A in BD or BE.

NOTE. The above analysis is also useful when, in the data, A is given instead of $B \sim C$.

(2) *To construct a triangle given two sides b, c and the median bisecting the third side.*

Analysis: Suppose ABC is the required $\triangle$. Bisect BC at D. Join AD and produce it to E making DE = AD. Join EC, EB.

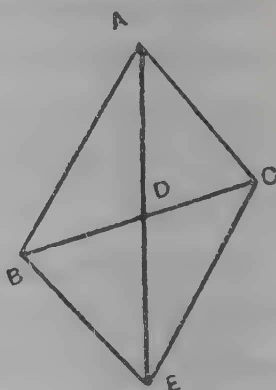


FIG. 124.

It is easily proved that ACEB is a parallelogram and EC is equal and parallel to AB.

In the $\triangle$ ACE, AC, CE and AE (= 2 AD) are known and therefore the $\triangle$ ACE can be determined and hence, by completing the parallelogram ACEB, the $\triangle$ ABC is also known.

NOTE. For the solution to be possible, the $\triangle$ ACE must be possible. $\therefore \quad AB \sim AC < 2AD < AB + AC.$

(3) *To construct a triangle given either two medians and a side or three medians.*

Analysis: Suppose ABC is the required $\triangle$. Bisect AC, AB at D, E. Join BD, CE and let them intersect at G.

We shall now show that BG = 2 GD, CG = 2 GE.

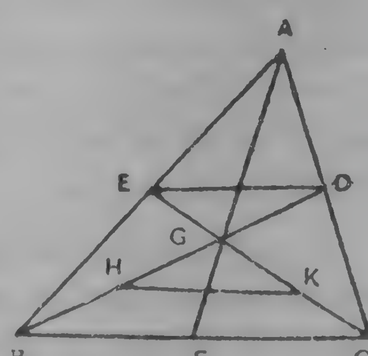


FIG. 125.

Bisect BG, GC at H, K respectively.

It is easily seen that ED is parallel to BC and half of it and HK is parallel to BC and half of it.

$\therefore$ ED and HK are equal and parallel.

$\therefore$ EDKH is a ▭m and the diagonals EK, HD bisect each other at G; so that

$$BG = 2GH = 2GD \text{ and } CG = 2KG = 2GE.$$

Thus the median CE passes through a point G in BD such that $BG = 2GD$ and $CG = 2GE$.

Similarly it can be shown that the median AF bisecting BC at F passes through the same point G such that $AG = 2GF$.

Thus, the medians of a triangle are concurrent and the point of concurrence (known as the **centroid**) *divides each of them in the ratio* 2 : 1, *the greater segments being towards the vertices.*

The disposition of the medians within the $\triangle$ ABC which the above analysis reveals, gives the clue to the construction of the $\triangle$ ABC, given some sides and medians.

Thus, to get the $\triangle$, given two medians (say BD, CE) and a side BC, we may first construct the $\triangle$ BGC in which $BG = \frac{2}{3}BD$, $GC = \frac{2}{3}CE$, and BC are known. From the $\triangle$ BGC to the $\triangle$ BAC is an obvious step.

To construct the $\triangle$ ABC given BD, CE and AC, we should first construct the $\triangle$ GDC in which $GD = \frac{1}{3}BD$, $GC = \frac{2}{3}CE$ and $DC = \frac{1}{2}AC$. By producing DG to B and CD to A so that BD and CA are of given lengths, the $\triangle$ ABC is determined.

To construct the $\triangle$ ABC, given three medians, is easily seen to reduce itself to the construction of the $\triangle$BGC in which $BG = \frac{2}{3}BD$, $GC = \frac{2}{3}CE$ and the median GF (of the $\triangle$ BGC) $= \frac{1}{3}AF$, are known. Since two sides and a

median of the $\triangle BGC$ are known, the $\triangle$ can be constructed by (2) above and hence the $\triangle ABC$ derived.

Cor.: $BG + GC > 2GF$, *i.e.*, $> AG$.

Hence, the sum of any two medians of a triangle is greater than the third.

EXERCISE XXVI.

Construct triangles to satisfy the following data and wherever the solution fails or the triangles are impossible, point out the discrepancy in the data. Show also how one of the data may be modified to make the solution possible. Wherever there is apparently more than one solution, indicate whether the solutions are congruent or not.

1. $a = 7{\cdot}3$ cm., $b = 10{\cdot}7$ cm., $C = 100°$.
2. $a = 4{\cdot}6$ in., $c = 5{\cdot}9$ in., $B = 74°$.
3. $A = 70°$, $B = 110°$, $a = 4''$.
4. $B = 100°$, $C = 85°$, $b = 3{\cdot}9''$.
5. $A = 90°$, $B = 65°$, $a = 9{\cdot}7$ cm.
6. $A = 73°$, $B = 92°$, $c = 1''$.
7. $a = 7$ cm., $b = 8$ cm., $c = 15$ cm.
8. $a = 5$ cm., $b = 12$ cm., $c = 13$ cm.
9. $a = 7{\cdot}2$ in., $b = 6{\cdot}4$ in., $c = 10{\cdot}6$ in.
10. $B = 70°$, $c = 6''$, $b = 5''$.
11. $C = 30°$, $a = 7{\cdot}2$ cm., $c = 3{\cdot}6$ cm.
12. $A = 40°$, $a = b = 1{\cdot}7''$.
13. $A = 50°$, $a = 8{\cdot}5$ cm., $b = 10$ cm.
14. $C = 138°$, $b = 6$ cm., $c = 10$ cm.
15. $B = 100°$, $b = 3$ in., $a = 4$ in.
16. $B = 110°$, $a = 10$ cm., $b = 9{\cdot}7$ cm.

17. A = 136°, $a = c = 6{\cdot}3$ cm.

18. C = 90°, $c = 7{\cdot}9$ in., $b = 8{\cdot}3$ in.

19. A = 90°, $a = 10$ in., $b = 7{\cdot}8$ in.

20. A = 60°, B = 80°, $a + b = 15$ cm.

21. B = 85°, C = 73°, $b - a = 1$ in.

22. C = 50°, A = 65°, $a - b = 10$ in.

23. A = 70°, B = 52°, $a+b+c = 30$ cm.

24. B = 90°, C = 40°, $a+b-c = 2$ in.

25. C = 60°, A = 25°, $b+c-a = 5$ cm.

26. A = 2B = 72°, $2c - b = 2{\cdot}5$ in.

27. $b + c = 17$ cm., $a = 8$ cm., B — C = 25°.

28. $c - a = 1$ in., $b = 3{\cdot}4$ in., C — A = 20°.

29. $a + b = 3{\cdot}8$ in., $c = {\cdot}9$ in., A = 65°.

30. $b - c = {\cdot}5$ in., $a = 2$ in., B = 32°.

31. Construct an equilateral triangle given the sum of a side and an altitude to be equal to 4 inches.

32. Construct a square, given the difference between a side and a diagonal to be 2·5 cm.

33. Construct a rhombus given (i) one side and a diagonal; (ii) two diagonals; (iii) one diagonal and one angle; (iv) one side and the sum of the two diagonals; (v) one side and the difference of the diagonals.

34. Construct a trapezium ABCD given AB = 3″, BC = 5″, CD = 10″, DA = 6″, AB and CD being parallel.

35. Given four rods of lengths 10″, 8″, 5″ and 4″ in how many different ways can you place them so as to form a quadrilateral with two sides parallel?

36. ABC is a triangle. D, E, F are the middle points of the sides CA, AB, BC respectively. The medians meet

in G. Show how you will construct the triangle ABC from the following measurements and indicate the cases of failure, if any :

(i) $b = 8$ cm., $c = 12$ cm, AF $= 5$ cm.

(ii) $a = 2''$, $b = 3''$, BD $= 2{\cdot}5''$.

(iii) $b + c = 15$ cms., AF $= 6$ cms., $\angle$BAC $= 70^\circ$.

(iv) $a - b = 1$ in., $\angle$BCA $= 100^\circ$, CE $= 3$ in.

(v) BD $= 7$ cms., CE $= 10$ cms, $b = 12$ cms.

(vi) AF $= 4{\cdot}2$ cms., BD $= 9{\cdot}6$ cm, $c = 7{\cdot}8$ cms.

(vii) $b = 3{\cdot}6$ in., $c = 5{\cdot}8$ in., $\angle$FAC $-$ $\angle$FAB $= 20^\circ$.

(viii) AF $= 6{\cdot}7$ in., BD $= 3{\cdot}5$ in., CE $= 5{\cdot}9$ in.

(ix) $c - b = 12$ cms., $\angle$FAC $-$ $\angle$FAB $= 30^\circ$, AF $= 10$ cms.

(x) $\angle$BAF $= 20^\circ$, $\angle$FAC $= 40^\circ$, AB $+$ AC $+$ 2 AF $= 10$ in.

37. The bisector of the angle BAC of a triangle ABC meets BC in D. X is the foot of the perpendicular from A to BC. Prove that $\angle$DAX $= \frac{1}{2}$ (B $\sim$ C). Hence show that any two of the three data (AD, AX, B $\sim$ C) determine the third.

How will you construct the Δ ABC given the median and the altitude from A to BC and the difference of the angles at B and C ?

§ 4. Construction of Quadrilaterals.

1. Since a quadrilateral can be divided into two Δs by a diagonal (*vide* fig. 126), the construction of the quadrilateral depends upon the construction of these two triangles in succession. Now, the construction of the first triangle requires three conditions,

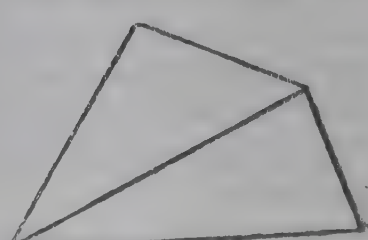


FIG. 126.

while the second triangle which has one element in

common with the first requires two conditions for its determination. Thus a quadrilateral requires, in all, five conditions.

Similarly, we may argue that the construction of a pentagon which can be considered to be made up of a quadrilateral and a triangle requires 5 + 2 or 7 conditions and so on.

2. Distributing the five necessary and sufficient conditions among the sides, diagonals and angles of a quadrilateral, we get the following different types of groups of data :

(i) 4 sides and 1 angle.

(ii) 4 sides and 1 diagonal.

(iii) 3 sides, 2 angles.

(iv) 3 sides, 2 diagonals.

(v) 3 sides, 1 angle, 1 diagonal.

(vi) 2 sides, 2 diagonals, 1 angle.

(vii) 2 sides, 1 diagonal, 2 angles.

(viii) 2 sides, 3 angles.

(ix) 1 side, 2 diagonals, 2 angles.

(x) 1 side, 1 diagonal, 3 angles.

(xi) 2 diagonals and 3 angles.

3. It is not within the scope of this book to discuss all the above cases in detail, some of which are easy, some difficult and sometimes not even within the range of Euclidean geometry. For instance, case (ii) is obvious, as soon as you draw a rough figure and mark the data therein, while case (xi) is beyond the reach of our elementary geometry.

We shall conclude this section by explaining a useful step in the analysis of many of the cases of construction of quadrilaterals where the data include angles.

Construction 11.

To construct a quadrilateral ABCD, *given* AB, BC, CD *and the angles* BAD, ADC.

Analysis: Suppose ABCD is the required quadrilateral. Complete the ▭m BADE.*

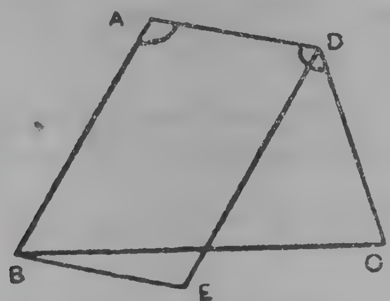


FIG. 127.

In the $\triangle$ EDC†, ED = AB, $\angle$EDC = $\angle$ADC — $\angle$ADE = $\angle$ADC + $\angle$BAD — 180° and therefore known.

Hence the $\triangle$ EDC can be constructed. Since $\angle$ADC is known, the direction of AD is known.

Since EB is parallel to AD and BC is of known length, the position of B is obtained as the intersection of the parallel through E to AD and the arc with centre C and a known radius. The parallel through B to ED meets DA in A and the quadrilateral is thus completely determined.

NOTE: The above analysis is also useful in the case where 4 sides and the angle between a pair of opposite sides are given and in the case where a pair of opposite sides, a pair of diagonals, and the inclination between the diagonals are given.

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* This is the useful step referred to.

† EC is not joined in the fig.

EXERCISE XXVII.

Construct quadrilaterals ABCD to satisfy the following data; if in any case the quadrilaterals are impossible, state the reason for the failure of solution and show how one or more of the data may be modified for the solution to be possible:

1. AB = 8 cms., BC = 7 cms., CD = 6 cms., DA = 4 cms., $\angle$BAD = 90°.

2. AB = 3·5″, BC = 4″, CD = 5·3″, DA = 2″, the angle between AB and CD is 50°.

3. AB = 10 cms., BC = 4 cms., DA = 6 cms., C = D = 80°.

4. AB = 4 in., CD = 2 in., DA = 5 in., B = C = 90°.

5. AB = BC = 3 in., CD = 2 DA = 4 in., AC = 4 in.

6. AB = 2 in., BC = 3·5 in., AD = DC, $\angle$ABC = 2 $\angle$ADC = 80°.

7. AB = 5 cms., BC = 6 cms., CA = 7 cms., BD = 8 cms., DC = 9 cms.

8. AB = 7·5 cms., BC = 10 cms., CD = 5·7 cms., the angle between AB and CD is 43°; the angle between DA and CB is 30°.

9. $\angle$BAD = 70°, AB = BC = CD = BD = 2 in.

10. $\angle$ABC = 180° — $\angle$BCD = $\angle$ADC, AC = 2 BD = 8·6 cms., the angle between the diagonals is 75°.

11. $\angle$ABC + $\angle$BCD = 180°, AB = $\frac{1}{6}$ CD = $\frac{1}{3}$ BC = $\frac{1}{4}$ AD = ·3″.

12. $\angle$BAD = 100°, $\angle$BCD = 75° = $\angle$ABC, AB = $\frac{1}{2}$ CD = 1·4″.

13. $AB = 1{\cdot}3$ in., $BD = 2{\cdot}5$ in., $DC = 1$ in., $CA = 2$ in., the angle between AC and BD is $30°$.

14. $AB = 7{\cdot}8$ cms., $AC = 5{\cdot}3$ cms., $\angle BAD = \angle ADC = \angle DCB = 110°$.

15. Construct a quadrilateral, given :

(i) the four sides and the length of the straight line joining the middle points of one pair of opposite sides.

(ii) the lengths of two opposite sides and the middle points of three sides.

CHAPTER IX.

LOCI.

§ 1. Definitions and Preliminary Notions.

1. If we look at the centre of a wheel rolling along a road, we find that it occupies different positions at different times and describes a curve in space parallel to the trace of the wheel on the road.

Similarly, you find the tips of the hour, minute, and second hands of a watch tracing circles along the rim of the watch-dial in the course of 12 hours, 1 hour and 1 minute respectively.

Again, if you observe the stars in the sky, you will notice how they change their positions in the course of a day and appear to move along arcs of circles from west to east. The Sun and the Moon also trace their own paths in the sky from day to day. One note-worthy feature about these paths is that they are traced not in an arbitrary manner, but subject to certain laws. In Geometry, we give the name 'locus' to a path traced by a point moving according to a given law or condition.

Def.: **A locus is the totality of paths actually traced by a point moving according to a given law.**

NOTE: The word '*totality*' implies that every point that satisfies the given condition lies on the path; and the words '*actually traced*' imply that every point that lies on the path has actually been one of the positions of the moving point.

The locus, therefore, has a double aspect:

(i) **inclusive**, *i. e.*, including every point which satisfies the given condition, and

(ii) **exclusive**, *i. e.*, excluding every point which does not satisfy the given condition.

It is important to notice that a locus-theorem is in fact a double theorem, a theorem and its converse, though in the great majority of cases, one of the two theorems can be easily inferred from the other by the method of *Reductio ad absurdum* or by merely reversing the steps of proof.

2. The locus of a point is like the address of a person, and if we know the locus on which a point should lie, we know where to search for it. If a point should satisfy two conditions, we may find the two loci corresponding to the conditions and fix the position of the point where the two loci intersect. Thus a knowledge of loci is helpful in construction-problems.

Example: To find a point at a distance of 1″ from a fixed point O and ·5″ from a fixed straight line AB.

The locus of the point determined by the first condition is the circle with centre O and radius 1″; while the locus of the same point determined from the other condition is a pair of straight lines parallel to AB at a distance from it equal to ·5″. The four points P, P′, Q, Q′ which are the intersections of these two loci are the different possible positions of the required point.

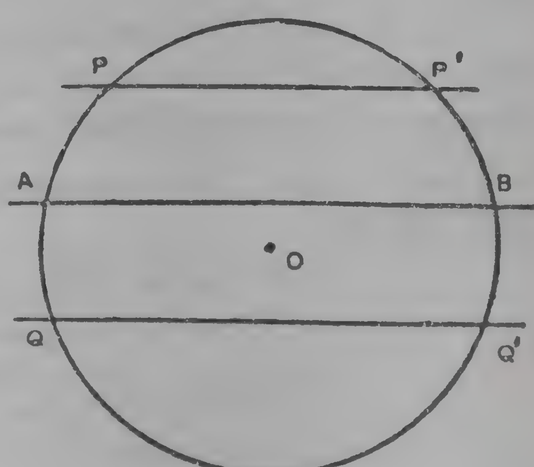

FIG. 128.

§ 2. Two Important Loci.

Theorem 19.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

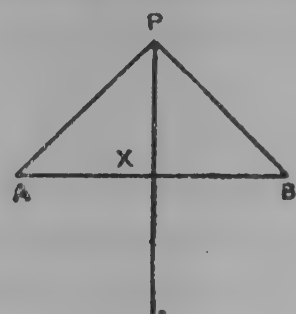

FIG. 129.

Let A, B be two fixed points and P any point such that PA = PB.

It is required to prove that the locus of P is the perpendicular bisector of AB:

i. e., (i) that P lies on the perpendicular bisector of AB, and conversely (ii) that every point on the perpendicular bisector of AB is equidistant from A and B.

Proof: Let X be the middle point of AB. Join PA, PB, PX.

In the $\triangle$s PXA, PXB

$$\begin{cases} PA = PB \text{ (given)} \\ XA = XB \text{ (const.)} \\ PX \text{ is common.} \end{cases}$$

$\therefore$ $\triangle PXA \equiv \triangle PXB$, so that

$\angle PXA = \angle PXB$.

Hence PX is perpendicular to AB.

$\therefore$ P lies on the perpendicular bisector of AB ... (1)

Again, without loss of generality, we may denote by P any point on the perpendicular bisector of AB.

Join PA, PB

In the $\triangle$s PXA, PXB.

$$\begin{cases} XA = XB \\ PX \text{ is common} \\ \angle PXA = \angle PXB \text{ (rt. } \angle\text{s.)} \end{cases}$$

$\therefore$ $\triangle PXA \equiv \triangle PXB$, so that

$PA = PB$.

Thus, any point P on the perpendicular bisector of AB is equidistant from A and B ... ... (2)

From (1) and (2), we conclude that the locus of P is the perpendicular bisector of AB.

NOTE. If the locus is not confined to a plane passing through AB, *i. e*, if P could move anywhere in space subject to the condition that PA = PB, the locus is the plane generated by revolving XP through 4 rt. $\angle$s about AB as axis.

Theorem 20.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

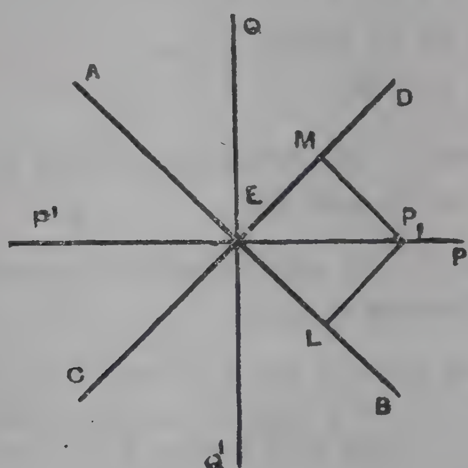

FIG. 130.

Let AB, CD be two straight lines intersecting at E and PP′, QQ′ the bisectors† of the angles between them.

Let P_1 be any point such that the perpendiculars P_1 L, P_1 M on AB, CD respectively, are equal. It is required to prove that the locus of P_1 is the pair of straight lines PP′, QQ′:

i. e., (i) that P_1 lies on one of the two lines PP′, QQ′. and (ii) that every point on these lines is equidistant from AB and CD.

---

† The bisectors of vertically opposite angles are in the same straight line. (*Vide* p. 57. Ex. 5.)

Proof: Let P_1 lie within $\angle$ BED. Join P_1E.

In the rt. $\angle$d $\triangle$s P_1LE and P_1ME.

$$\begin{cases} P_1L = P_1M \\ \text{and the hypotenuse } P_1E \text{ is common.} \end{cases}$$

$\therefore \quad \triangle P_1LE \equiv \triangle P_1ME$, so that

$\angle P_1EL = \angle P_1EM$ *i.e.*, P_1 lies on the bisector PP'

Similarly, we can show that if P_1 lies anywhere else, it must be on one or other of the bisectors of the angles between the straight lines AB and CD ... (1)

Again, without loss of generality, we may denote by P_1 any point on one of the lines PP' and QQ'.

From P_1 draw P_1L, P_1M perpendiculars to AB, CD respectively. In the $\triangle$s P_1LE and P_1ME

$$\begin{cases} \angle P_1LE = \angle P_1ME \text{ (rt. } \angle\text{s)} \\ \angle P_1EL = \angle P_1EM \text{ (since } P_1E \text{ is a bisector)} \\ P_1E \text{ is common.} \end{cases}$$

$\therefore \quad \triangle P_1LE \equiv \triangle P_1ME$, so that

$P_1L = P_1M$ *i. e*, P_1 is equidistant from AB, CD. (2)

From (1) and (2), we conclude that the locus of P is the pair of bisectors PP', QQ'.

NOTE 1. If AB and CD be parallel, the locus of a point equidistant from the two straight lines is the straight line parallel to them and mid-way between them.

NOTE 2. The locus of a point equidistant from two intersecting planes is the pair of planes bisecting the angle between the two planes.

The locus of a point equidistant from two given parallel planes is another parallel plane lying midway between the given planes.

EXAMPLE 1. *The perpendicular bisectors of the sides of a triangle are concurrent.*

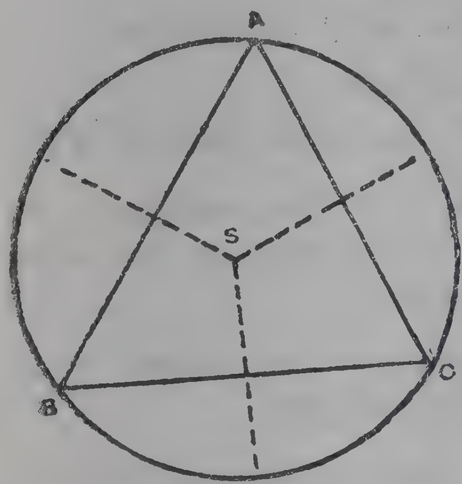

FIG. 131.

Let ABC be a triangle and the perpendicular bisectors of AB, AC meet in S. It is required to prove that the perpendicular bisector of BC also passes through S.

Proof: Since S is a point on the perpendicular bisector of AB, which is the locus of points equidistant from A and B, it follows that SA = SB.

Similarly, SA = SC.

∴ SB = SC, *i. e.*, S is a point which is equidistant from B and C.

∴ S lies on the perpendicular bisector of BC, which is the locus of points equidistant from B and C.

∴ The perpendicular bisectors of the sides of triangle ABC are concurrent.

NOTE 1. Since SA = SB = SC, the circle with centre S and radius SA will pass through B and C. This circle is called the **circum-circle** of the $\triangle$ ABC; its centre S is the **circum-centre** and its radius the **circum-radius**.

NOTE 2. When $\angle BAC = 90°$, S coincides with the middle point of BC.

EXAMPLE 2. *The internal bisectors of the angles of a triangle are concurrent; the external bisectors of two angles and the internal bisector of the third angle are concurrent.*

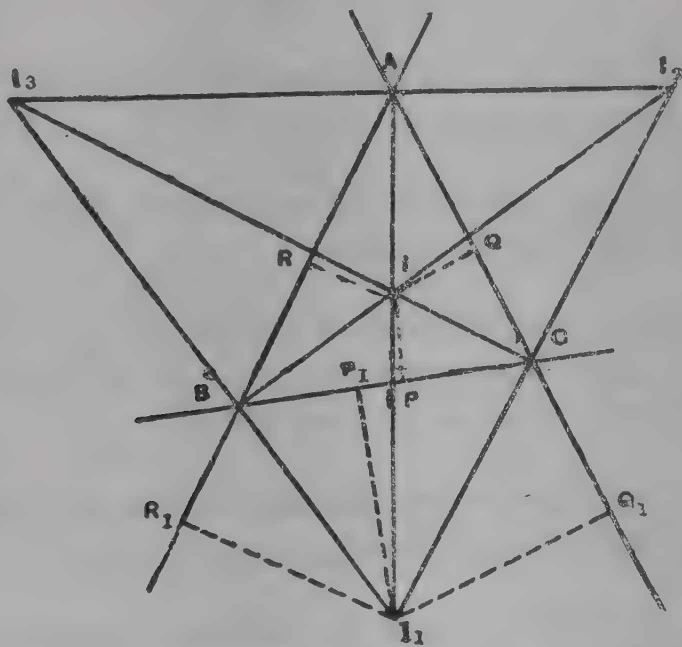

FIG. 132.

Let ABC be a triangle and the internal bisectors of the angles at B and C meet at I, while the external bisectors meet at I_1. It is required to prove that the internal bisector of $\angle$ A passes through I and I_1.

Construction: From I draw IP, IQ, IR perpendiculars to BC, CA, AB respectively. From I_1 also draw I_1P_1, I_1Q_1, I_1R_1 perpendiculars to the sides.

Proof: Since I is a point on the internal bisector of the angle between AB and BC, I is equidistant from AB and BC.

$\therefore$ IR = IP

Similarly, IQ = IP.

$\therefore$ IQ = IR ... ... (i)

Again, since I_1 lies on the external bisector of the angle between AB, BC and also on the external bisector of the angle between BC, CA, we prove as before

$$I_1Q_1 = I_1R_1 \quad \dots \quad \dots \text{(ii)}$$

From (i) and (ii), we infer that I and I_1 must lie on a bisector of the angle between AB, AC.

Since I and I_1 lie within the angle BAC, they must lie on the internal bisector of angle A.

$\therefore$ The internal bisectors of the angles A, B, C are concurrent, as also the external bisectors of the angles B, C and the internal bisector of the angle A.

Cor.: Let the external bisectors of the angles A, B, C form a triangle with vertices I_1, I_2, I_3.

Then AI_1, BI_2, CI_3 are the internal bisectors of the angles A, B, C and perpendicular respectively to the external bisectors of the same angles, *viz*, I_2I_3, I_3I_1 and I_1I_2.

$\therefore$ AI_1, BI_2, CI_3 are the altitudes of the $\triangle I_1I_2I_3$ and meet at I.

I is called the **in-centre** of the $\triangle$ ABC, I_1, I_2, I_3 are called the **ex-centres**; and all these four points are equidistant from AB, BC, CA, the sides of the triangle ABC.

EXAMPLE 3. *XOX′, YOY′ are two straight lines intersecting at right angles at O. P is a point which moves so that the sum of its distances from XX′, YY′ is constant. Find the locus of P.*

Construction: From P draw PL, PM perpendicular to XX′, YY′ to meet them at L and M respectively.

On OX, OX′, OY, OY′ take lengths OQ, OQ′, OR, OR′ respectively, each equal to the constant length PL + PM. Then, evidently, Q, Q′, R, R′ are fixed points. Join QR, RQ′, Q′R′, R′Q.

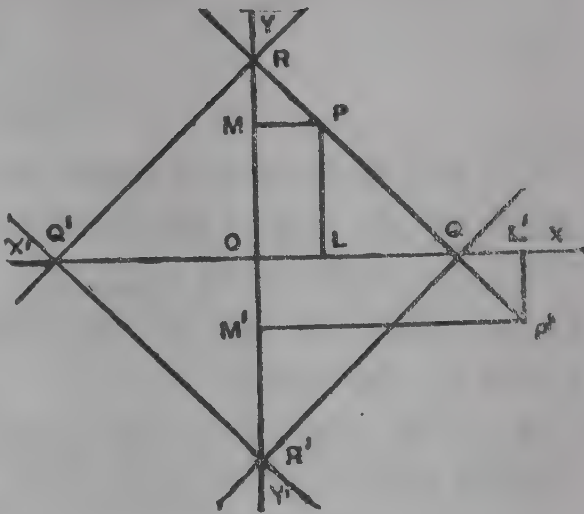

FIG. 133.

We shall now show that the four finite straight lines QR, RQ′, Q′R′, R′Q constitute the required locus.

Proof: The fig. OLPM is a rectangle, since the opposite sides are parallel and the angle at O is right.

$\therefore$ PL = OM and PM = OL (being opposite sides).

Now if P lies in the quadrant XOY,

$$OQ = PL + PM$$
$$= PL + OL.$$

$\therefore$ $PL = OQ - OL = LQ.$

Since $\angle PLQ$ is a right angle, if we join PQ,

$$\angle PQL = \angle LPQ = 45^\circ.$$

In the Δ OQR, since OQ = OR, $\angle OQR = \angle ORQ$,

$\therefore$ $\angle OQR = 45^\circ$.

$\therefore$ $\angle OQR = \angle PQL$.

$\therefore$ P lies on the straight line QR.

Similarly, we can show that

if P is in the quadrant YOX', P lies on $Q'R$;

„ „ $X'OY'$, „ $Q'R'$;

„ „ $Y'OX$, „ $R'Q$.

∴ P lies on one or other of the four straight lines which are the sides of the quadrilateral $QRQ'R'$.

Next, without loss of generality, we can suppose that P is any point on one of the sides of the quadrilateral $QRQ'R'$, say on QR.

From P, draw PL, PM perpendicular to XX', YY' meeting them in L, M respectively.

As before, OLPM is a rectangle and ∴ $PM = OL$.

Again, in the $\triangle PLQ$,

$$\angle PLQ = 90^\circ \text{ and } \angle PQL = 45^\circ.$$

$$\therefore \quad \angle LPQ = 45^\circ = \angle PQL.$$

$$\therefore \quad LQ = PL.$$

Hence $PL + PM = LQ + OL = OQ$.

∴ $PL + PM$ is constant.

Therefore, the locus of P is the four sides of the quadrilateral $QRQ'R'$.

NOTE 1. The quadrilateral $QRQ'R'$ is a square.

NOTE 2. The points on QR, RQ', etc., produced either way do not satisfy the required condition, but some other condition.

For, let P' be any point on RQ produced. From P' draw $P'L'$, $P'M'$ perpendicular to XX', YY'.

It is easily shown that $P'L' = L'Q$ and $P'M' = L'O$.

$$\therefore \quad P'L' \sim P'M' = OQ,$$

i.e., the difference of the distances of P′ from XX′, YY′ is constant. Conversely, we can show that if P′ is such that P′L′ ∼ P′M′ is constant, P′ lies on one of the sides produced, of the square QRQ′R′.

Thus, the produced parts of the sides of the square QRQ′R′ constitute the locus of a set of points satisfying a different condition, *viz.*, the difference of the distances of the points from XX′, YY′ is constant.

NOTE 3. This example illustrates that the 'locus' in Geometry is the counterpart of 'graph' in Algebra; and the geometrical condition a point on the locus satisfies corresponds to the (algebraic) equation of the graph.

One way of representing a point in a plane is by giving its distances from two fixed straight lines in the plane cutting one another at right angles. By appropriate conventions for these distances, it is possible to define a point uniquely in the plane with reference to these fixed lines called axes of co-ordinates. The two distances are called the co-ordinates and denoted by the letters x, y and any condition which these distances satisfy is expressed algebraically by an equation.

Thus, the geometrical condition in the above example can be represented algebraically by the equation $x + y = a$ where a is a constant. The graph of this equation is obtained by plotting the points whose co-ordinates satisfy the equation and joining them by a smooth line.

Since the conventions regarding signs are not ordinarily followed in geometry, the graph of $x + y = a$ does not exactly coincide with the geometrical locus. For the geometrical locus and the algebraic graph to agree with one another, we must adopt the same conventions regarding signs in both geometry and algebra.

EXERCISE XXVIII.

1. A coin is rolled along a circular groove. What is the locus traced by its centre?

2. What is the locus traced by a corner of a set-square when the opposite side moves on an edge of your paper?

3. A man walks in the middle of a road keeping always the same distance from each side of the road. What is his locus?

4. A point moves inside an equilateral triangle always keeping the same distance from the sides. What is the locus traced by the point?

5. A fruit falls from a branch of a tree. What is the locus described by the fruit?

6. ABC is a vertical arc of a circle. If chains of equal lengths are hung from every point on this arc, what is the locus of the free extremities of the chains?

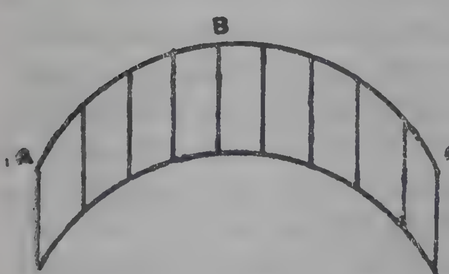

FIG. 134.

7. An aeroplane flies horizontally at a height of half a mile so as to be equidistant from two cities. What is the locus described by it?

8. A horse races round a circular track. A man moves so as always to be at the same distance from the horse and the centre of the track. What locus does the man describe?

9. A body moves on the earth's surface so as always to be equidistant from two places on the surface. What is the locus of the body?

10. An ant moves on the curved surface of a vertical cone (standing on a circular base) so as always to be equidistant from the vertex of the cone and its circular base. How will you determine the locus traced by the ant?

11. BC is a finite straight line. A is a point which moves so that $\angle ABC = \angle ACB$. What is the locus of A? Is the middle point of BC included in the locus?

12. BC is a finite straight line. A point P moves so that PB is always at right angles to PC. Show that the locus of P is a circle.

13. A pencil slides in a vertical plane between a wall and a table whose top is at right angles to the wall. What is the locus described by the middle point of the pencil?

14. A number of (1) rectangles, (2) rhombuses are constructed on a given base. What is the locus of their centres?

15. (i) P is a fixed point within a fixed circle, whose centre is O. Q is any point on the circle. What is the locus of the middle point of PQ?

(ii) What is the locus of the middle point of PQ if Q moves on a fixed straight line, P being fixed in position?

16. P is a fixed point outside a given circle. What is the locus of the middle points of chords passing through P?

17. Given the base and the length of the altitude perpendicular to the base, what is the locus of the opposite vertex?

18. Given the base in position (but not in magnitude) the difference of the base angles, and the length of the internal bisector of the vertical angle, what is the locus of the opposite vertex?

19. What is the locus of the middle points of chords of given length placed within a fixed circle? What does the locus become when the given length is equal to a diameter of the circle?

20. ABC is a Δ. Find points D, E in AB, AC (or AB, AC produced) which are equidistant from every point in BC.

21. What is the locus of a point which moves so that the difference of its distances from two given intersecting straight lines (not at right angles to each other) is constant?

What does the locus become when the difference is zero? What does the locus become when the given straight lines are parallel?

22. Find pairs of points which are equidistant from two sides of a triangle and also (equidistant) from the vertices opposite to these sides. Show that these points lie on the circum-circle of the triangle.

23. Given a straight line AB, and two points P, Q, draw through P and Q two straight lines PR, QR such that every point in the straight line AB or AB produced is equidistant from PR and QR. Examine the particular cases where P and Q are equidistant from AB.

24. Find a point which is equidistant from two given points A, B and also from two given straight lines OX, OY. How many such points are there? What happens when OX, OY are equally inclined to AB?

25. A treasure is buried in a garden at a place which is at the same distance from two trees in the garden and equidistant from two adjacent compound walls. The plan of the garden is shown in the figure. Locate the possible positions of the treasure.

FIG. 135.

26. Find a point at given distances from two given intersecting straight lines. How many such points are there?

27. Find points which are equidistant from

(i) three straight lines, no two of which are parallel;

(ii) ,, two of which are parallel.

Are there points equidistant from (i) three concurrent straight lines, (ii) three parallel straight lines, (iii) four straight lines, no two of which are parallel?

28. Find a point which is at a given distance p from a given point and also at a given distance q from a given straight line. Discuss the cases where there are (i) four, (ii) three, (iii) two points, (iv) one point and (v) none satisfying the given condition. Illustrate the cases by taking numerical values for p and q.

29. * Plot the locus of a point which moves so that

(i) it is equidistant from a given point and a given straight line;

(ii) the difference of its distances from two given points is constant;

(iii) the sum of its distances from two given points is constant;

(iv) the product of its distances from two given straight lines at right angles to each other is constant;

(v) its distance from one given point is twice its distance from another given point;

(vi) its distance from a given point is twice its distance from a given straight line;

(vii) its distance from a given point is half its distance from a given straight line;

(viii) its distance from one given straight line is twice its distance from another given straight line.

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* Just as, for drawing the graph of an equation, we plot several points which satisfy the equation, we may also plot several points which satisfy a geometrical condition and arrive at the shape of the locus practically. This is called '**plotting the locus**.'

REVISION PAPERS.

Paper I.

1. In a triangle ABC, $\angle A - \angle B = 72^\circ$, $\angle B - \angle C = 36^\circ$. Find the angles of the triangle and prove that if an equilateral $\triangle$ BDC be described on the base BC, opposite to A, AD bisects the angle BAC.

2. Can you draw a curve, other than a circle, such that every point on the curve is at the same distance from a fixed point not on the curve? If there should be such a curve, on what kind of surface can it lie?

3. What sort of solid can be constructed from a figure of the shape shown in the margin by folding it about the dotted lines?

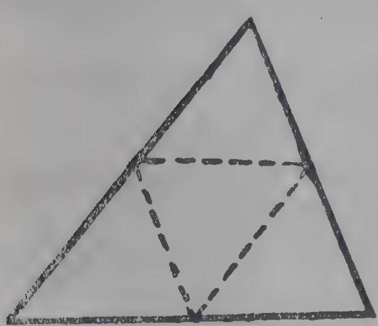

FIG. 136.

Show that, in the solid so obtained, the opposite pairs of edges are equal.

4. Construct a trapezium ABCD, in which AB is parallel to CD, given AD = 1 in., DC = 3 in., CB = 2 in., $\angle C = 30^\circ$.

Paper II.

1. ABC, DEF are two triangles in which $\angle A = \angle D$ and $\angle B = \angle E$. If AB = EF, are the two triangles congruent? Give reasons for your answer.

2. If two parallel planes are cut by a third plane, show that the lines of intersection are parallel.

3. What is the test of symmetry for plane figures? If a plane figure has *only* two axes of symmetry and no more, show that they must be at right angles.

4. Through a point P within a given angle AOB, draw a straight line terminated by OA, OB at Q, R respectively such that $PQ = 2PR$.

Paper III.

1. ABC is a triangle in which $AB = 2AC$ and $\angle ABC = 30^\circ$. Show that the angle at C is a right angle.

2. Three straight lines OA, OB, OC meet at O such that $\angle AOB + \angle BOC + \angle COA = 360^\circ$. Show practically, that OA, OB, OC must lie in the same plane.

3. A man walks 100 yds. first in the direction N.N.E, and then the same distance successively in the directions N.N.W, W.N.W, and W.S.W. How far is he from his starting point? In what direction should he walk finally to reach the starting point?

4. Show that it is impossible to construct a triangle, given the base $6''$, the difference of the base angles 90° and the difference of the other two sides $4''$.

Paper IV.

1. OA, OB are two intersecting straight lines. P is a point within the $\angle AOB$ such that the perpendicular from P to OA is greater than the perpendicular from P to OB. Prove that $\angle POA > \angle POB$.

2. A pyramid stands on a horizontal plane on an equilateral triangular base. Two of its faces are vertical. Show that the remaining face is an isosceles triangle. Draw a net of the pyramid.

3. Prove the following test of parallelism of two straight lines AB, CD:

Take two equal lengths PQ, RS on the two straight lines AB, CD respectively. Join PR, QS towards the same parts.

If PR = QS, then the two straight lines are parallel.

If PR > QS, then PQ, RS will meet on the side of Q, S.

If PR < QS, then PQ, RS meet on the side of P, R.

4. Construct a square on a side of 1″. Inscribe within it an equilateral triangle having one vertex at a corner of the square.

Paper V.

1. ABC is a triangle in which AB = 2AC. The external bisector of the angle BAC meets BC produced in D. Prove that BC = CD.

2. Show that a section of a sphere by a plane is a circle.

3. Draw an equilateral triangle on a side of 2″. Inscribe within it a regular hexagon, with three alternate sides along the sides of the triangle.

4. Quote two theorems in geometry whose converses are (i) true, (ii) not true.

Is the converse of the following theorem true?—

'If O is a point within a triangle ABC, then

$$OA + OB + OC < AB + AC + BC.'$$

Paper VI.

1. If a vertical angle of a triangle is contained by two unequal sides, and the median, the bisector of the angle and the altitude from the vertex to the opposite side be drawn, prove that the bisector lies intermediate in position and magnitude between the median and the altitude.

What happens when the sides including the vertical angle are equal?

2. A finite straight line OB moves about another straight line OA such that the angle AOB remains constant. What kind of surface does OB generate? What is the locus of B?

3. Show that it is not possible to divide a pentagon into less than three triangles and show also that it is possible to divide it into three triangles only in five different ways. State corresponding results for a polygon of n sides.

4. AB, CD are two given parallel straight lines $1''$ apart and P is a fixed point at a distance of $1''$ from AB and $2''$ from CD. Draw through P two equal straight lines PQ, PR at right angles to each other and terminated by AB, CD at Q, R respectively.

Paper VII.

1. In any triangle, prove that the sum of the medians is less than the perimeter but greater than three-fourths of the perimeter.

2. Why do the circles of latitude decrease as the latitude increases?

3. Among the theorems that you have studied you may have noticed that for some equality theorems there are corresponding inequality theorems. Quote a few such theorems.

Enunciate and prove the inequality theorems corresponding to the following theorem:

'If two sides of a triangle are equal, the medians bisecting them are also equal.'

4. PQRS is a billiard table. A ball is placed at a point A on the table at some distance from the corner Q and is so struck that, after hitting the cushions QP, PS, SR in succession, it falls into a pouch at the middle point of QR. Construct the path of the ball.

Paper VIII.

1. Q is any point on the external bisector of an angle APB. Prove that $AQ + QB > AP + PB$.

2. ABCD is a square. Find a point P in BC such that $AP = 2PC$.

3. ABC is a triangle cut out of card-board. It is folded about a straight line parallel to BC so that A falls on BC; and each of the remaining portions of the triangle is also folded so that B and C coincide with the position of A in BC (after folding) Show how these foldings can be effected and what properties of the triangle are revealed by this experiment Does this experiment ever fail?

4. The medians bisecting the sides AB, AC of a triangle ABC are 6 cm. and 9 cm. respectively. The length of the perpendicular from A to BC is 4·5 cm. Construct the triangle. How many such triangles are possible?

Paper IX.

1. ABC is an isosceles triangle in which $AB = AC$. If $AB \nless 2BC$, prove that $\angle BAC < 36°$.

2 A is a fixed point and P a variable point in the straight line BC. PQ is drawn at right angles to AP and equal to it. Find the locus of Q.

3. ABC, DEF are two triangles in which $AB = DE$. $AC = DF$, $AB > AC$ and $\angle BAC > \angle EDF$. Prove that the difference of the angles at B and C is greater than, equal to, or less than the angle EDF according as $\angle ABC$ is greater than, equal to or less than $\angle DEF$.

4. Construct a quadrilateral ABCD, given $AB = AC = CD = 1$ in., $\angle BAD = 90°$, and $\angle BCD = 165°$.

Paper X.

1. G is the centroid of a triangle ABC. If perpendiculars from the vertices of the triangle ABC be drawn on any straight line through G, prove that one of the three perpendiculars is equal to the sum of the other two.

2. ABC, DEF are two triangles in which $AB = DE$ and $AC = DF$. If $\angle A - \angle B = \angle D - \angle E$, prove that either the triangles are congruent or $\angle F + \angle C > 180°$.

Hence deduce that if the bisectors of the base angles of a triangle be equal, the triangle is isosceles.

3. ABC is an equilateral triangle. P is any point within it. Show that the sum of the distances of P from the sides of the triangle is constant.

4. A, B are two given points on the same side of a given straight line CD. Draw two straight lines AP, BQ equally inclined to CD, such that $AP \sim BQ$ or $AP + BQ$ is of known length.

ANSWERS.

BOOK I.

Exercise I. Pp. 19—22.

5. (i) 14; (ii) 35; triangle; pentagon; nonagon. **9.** A figure drawn on the surface of a sphere. **11.** (*a*) (i) 3; (ii) 4. (*b*) (i) 5; (ii) 20. **12.** (i), (iii), (iv), (v); prisms, (ii) pyramid.

15.

	S	F	E
(1)	4	4	6
(2)	8	6	12
(3)	8	6	12
(4)	6	8	12
(5)	12	20	30
(6)	30	12	30
(7)	6	6	10

18. Except the pentagon all the rest have central symmetry.

Exercise II. Pp. 33—42.

14. 5 times. **16.** (i) Obtuse, acute, obtuse; (ii) acute; (iii) one pair of opposite angles acute and another pair obtuse; (iv) two angles acute and one right; (v) right, acute, two of the angles of the triangular parallel faces must be acute and the angles of the parallelogram faces are either all right or one half of them acute and the rest obtuse. **17.** Yes. **18.** Four triangles; two right angles; eight right angles; yes. **20.** Six angles. **24.** 3·8 cms., the circles with radii 2, 3 satisfy (i); the circle with radius 4 satisfies (ii); the circles with radii 6, 7 satisfy (iii); the circle with radius 5 passes through B. **28.** No, since the four angles may be the four interior angles of the quadrilateral. **35.** 205 yds. **36.** $16\frac{7}{8}°$ E. of N. or $16\frac{7}{8}°$ W. of S. **37.** 115·5 ft. **38.** 70 ft. **39.** 2170 ft. **40.** 69° approx. **41.** Delete '300 ft. high.' 234 ft. **42.** 20°. **43.** 38 ft.: 213 ft.

Exercise III. Pp. 51-52.

1. $90°$. 2. $\angle ABC = 60°$, $\angle ABE = 30°$, $\angle ABD = 120°$. 3. $6°$. 4. $97\frac{1}{2}°$, $75°$, $52\frac{1}{2}°$, $150°$. 5. 4-30. 6. $120°$, $60°$, $70°$, $110°$. 7. $130°$, $50°$, $50°$. 8. (i) $45°$, (ii) $90°$, (iii) $45°$.

Exercise IV. Pp. 54-55.

8. $\angle COA = \angle BOD = 80°$, $\angle AOD = 100°$.

Exercise V. P. 57.

2. $\angle BOD = x°$, $\angle AOD = \angle BOC = 180° - x°$.

Exercise VI. Pp. 64-65.

10. No; for it is just possible that there are no parallel straight lines at all; in other words, all straight lines (in a plane) may intersect one another.

Exercise VII. Pp. 69-70.

2. $130°$, $50°$, $130°$. 4. $\angle PAB = 50°$, $\angle QAC = 60°$, $\angle BAC = 70°$. 8. BC need not be parallel to EF.

Exercise VIII. Pp. 73-74.

3. Right angle. Acute-angled triangle. 4. Yes, in an equilateral triangle. 14. (ii) $A = 113\frac{1}{3}°$, $B = 53\frac{1}{3}°$, $C = 13\frac{1}{3}°$. (iii) $\angle ADB = C + \frac{B}{2}$, $\angle AEC = B + \frac{C}{2}$ and so, if $\angle ADB = 2\angle AEC$. $\frac{B}{2} = 2B$ which is possible only when $B = 0$.

Exercise IX. Pp. 80—82.

1. 5 triangles; 10 right angles; 4 right angles. The sum of the angles of the triangles = the sum of the interior angles of the pentagon + the sum of the angles at the interior point. 2. Four triangles. The sum of the angles of the triangles = the sum of the interior angles. 5. (i) $156°$; (ii) $171°$; (iii) $176°$; (iv) $\left(2 - \frac{4}{n}\right)$ right

angles; (v) $\left(2-\frac{2}{n}\right)$ right angles; (vi) $\left(2-\frac{1}{n}\right)$ right angles. 6. (i) 5; (ii) 24; (iii) 25. 7. One of the angles is 180°. 8. 10 right angles, 14 right angles. 9. $n = 6$. 10. Either two squares or an equilateral triangle and a regular hexagon. 13. Triplets of polygons of sides (4, 5, 20), (4, 6, 12), (3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15). Of these combinations, only the second, *viz.*, (4, 6, 12) permits repetition and continuation since all the sides are even in number. 15. 11 sides.

Exercise X. Pp. 85—87.

1. 80°. 2. 125°. 3. $\frac{6}{7}$ right angle. $\frac{6}{7}$ right angle. $\frac{2}{7}$ right angle. 4. Either 50°, 80° or 65°, 65°. 5. 80° or 20°. 14. If BD = AD = BC, B = C = 72°; if BD = AD and BC = CD, B = C = $\frac{6}{7}$ right angle.

Exercise XI. P. 89.

1. (i) A > C > B, (ii) B > A > C, (iii) C = A > B. (iv) A > B > C.

Exercise XIII. Pp. 97-98.

1. The side opposite to the third angle. 6. If AB < AC and the interior bisectors of the angles B, C meet at E, then BE < CE. 9. If ∠BAC be acute, AD > $\frac{1}{2}$ BC.

Exercise XIV. Pp. 101-102.

1. The length of the third side must lie between 1″ and 9″. 2. (i) No; (ii) No. 8. OA + OB + OC + OD is least when O is the intersection of the diagonals. 14. The greatest is that which passes through the centres of both the circles; and the least is that which when produced passes through the centres.

Exercise XV. Pp. 109—111.

5. Produce AD to E making DE = AD. Join BE and prove AB = AE = 2AD. 13. Prove ∠OAB = ∠ODE

and hence $\angle OAC = \angle ODF$ and $\triangle OAC \equiv \triangle ODF$. **14.** With B as centre and BA as radius draw an arc and on it step off chords AD, DE, EC, each equal to the radius.

Exercise XVI. Pp. 114-115.

15. $DE = AD + CE$ or $AD - CE$ according as A, C are on the same side or on opposite sides of BD

Exercise XVII. Pp. 118-119.

1. Yes. **3.** No. **4.** Two, placed diagonally. **19.** Let O be the centre of the circle. Prove $\triangle OEB \equiv \triangle OED$ and hence $\angle OBA = \angle ODC$; next prove $\triangle OAB \equiv \triangle OCD$.

Exercise XVIII. Pp. 124-125.

5. (i) No; (ii) Yes. 6. No. 9. $\angle ABD = \angle ACD$.

Exercise XIX. Pp 128-129.

5. Yes. 6. The centre. 7. The three points coincide.

Exercise XX. Pp. 131—133.

2. $\angle AOF = \angle F'OB = 12\frac{1}{2}°$, $\angle FOF' = 15°$. The error in the angles AOF, F'OB is $6\frac{1}{4}$ % in defect and the error in the angle FOF' in $12\frac{1}{2}$ % in excess. **10.** Use the fact that $\angle ACB$ is right and show that $\triangle APG \equiv \triangle ACB$.

Exercise XXI. Pp. 135—137.

1. The altitudes meet at the vertex containing the right-angled triangle in (i), inside the triangle in (ii) and outside the triangle beyond the vertex containing the obtuse-angle. **6.** Through P draw straight lines parallel to the external and internal bisectors of the angle AOB. 8. Let AA' cut CD at L. Prove $\triangle APL \equiv \triangle A'PL$.

Exercise XXII. Pp. 139-140.

4. Draw parallels to AB, AC at a distance of 1″ from them and on both sides of the lines. **8.** Take a point T in QR and at T make $\angle QTA$ = the given angle. Draw PS parallel to the internal bisector of $\angle ATR$.

Exercise XXIII. Pp. 144—146.

1. Apply $\triangle A'B'C'$ to ABC so that $B'C'$ may coincide with BC and C' falls on the side of BC opposite to A. 2. *Vide* Note 1, pp. 122, 123. 5. If $AB > AE$, prove $AC > AD$ and $\angle ACB > \angle ADE$. Make $\angle ACF = \angle ADE$ so that F may lie between A and B. Prove $CF > DE$ and $BC > CF$. $\therefore BC > DE$. It follows that if $AB < AE$, $BC < DE$. If $AB = AE$, then $\triangle ABC \equiv \triangle AED$. $\therefore BC = DE$. From the Law of Converses, the required result is established. 6. Produce AD to E making $DE = AD$. Join EC. Compare $\triangle$s BAC, ECA. 8. Bisect AB at P and DE at Q. Join PC, QF. Compare the pairs of triangles (PCB, QFE), and (PCA, QFD). 10. No. $PB < PC$. 12. Produce BX, EY to A', D' respectively so that $XA' = XB$ and $YD' = YE$. Join $A'C$, $D'F$. Compare $\triangle$s $A'BC$, $D'EF$.

Exercise XXIV. Pp. 153—155.

10. No. In both the cases (i) and (ii), the quadrilateral is a parallelogram. 14. Produce OP to S making $PS = PO$ and draw SQ, SR parallel to OB, OA to form the parallelogram OQSR. Join QR. 15. Let ABC be the given triangle. Bisect $\angle B$ by BD meeting AC in D. Draw DE, DF parallel to BC, BA respectively to form the inscribed rhombus EBFD.

Exercise XXV. P. 160.

4. $EF = 11$ cms ; $CE = 8$ cms. 9. If ABCD be not a parallelogram, E, F, G, H are the vertices of a parallelogram; $EF < \frac{1}{2}(AP + CD)$ and $GH > \frac{1}{2}(AB \sim CD)$; the other results remain unaltered. 10. $1{\cdot}97''$, $2{\cdot}64''$, $3{\cdot}66''$, $4{\cdot}9o''$.

Exercise XXVI. Pp. 179—181.

3. No triangle is formed since two sides become parallel. 4. Two angles are together greater than two right-angles and hence no triangle. 7. The sum of two sides is equal to the third side and hence no proper triangle. 10. The perpendicular from A to BC is greater than b and

hence no solution. **13**. Two solutions are possible. **15 & 16**. No solution, since a lesser side cannot subtend a greater angle, **17 & 18**. No triangle **22**. Since $\angle B = 65^\circ$, $a = b$ and therefore $a - b \neq 10''$. **27 & 28**. Only one solution is possible. **29**. No solution. **35**. Denoting the lengths in descending order of magnitude by a, b, c, d, we note that for a trapezium with two sides, say, a, b, parallel, $c + d > a - b > c - d$. Such sets of conditions are possible only in three different ways; for example, if $c + d > a - b > c - d$, then $b + d > a - c > b - d$ and $c + b > a - d > b - c$. **36**. (iii) is a case of failure. **37**. Construct first a right-angled $\triangle$AFX with the given median as hypotenuse AF and the given perpendicular as the side AX Between AX and AF place AD so that $\angle$DAX = the given difference of the angles B and C. Draw AE perpendicular to AD. At F erect a perpendicular to cut AD, AE at L, M. Draw the circle with LM as diameter to cut FX produced at B, C. Join AB, AC. The solution fails when either AF < AX or $\angle$FAX < the given difference of the base angles.

Exercise XXVII. Pp. 184-185.

8. First construct $\triangle$ C′D′E′ such that C′D′ = 5·7 cms. D′E′ = 7·5 cms. and $\angle$D′ = 43°. Next, construct $\triangle$ BCE with BC = 10 cms., $\angle$CBE = 30° and CE = C′E′ and on EC construct $\triangle$ CDE $\equiv$ $\triangle$ C′D′E′. Complete the parallelogram BEDA. **10**. Use the property that the diagonals of a parallelogram bisect each other. **12**. The quadrilateral is impossible since AB < CD. This is a curious case where the construction does not fail, but the solution fails, being extraneous. **14**. There are two solutions for the $\triangle$ ABC but only one solution for the quadrilateral ABCD. **15**. (i) Let ABCD be the required quadrilateral, E, F, G the middle-points of AB, CD, AC. Then $\triangle$ EFG can be constructed. Produce EG to H making GH = EG and join CH. Now, the fig. EC is a ▭m, FC, CH being known, the position of C is fixed and hence A, B, D are easily determined. (ii) As before, we can fix the position of G the middle-point of AC. Since the

middle points of the sides of a quadrilateral are the vertices of a parallelogram, given three middle points, the fourth can be found out. Thus the middle points of the sides of the triangles ABC, ACD are known and hence the triangles can be constructed.

Exercise XXVIII.

1. A circle in space. **2.** A straight line. **3.** A straight line. **4.** The sides of another equil. $\triangle$ **5.** A vertical straight line. **6.** An equal vertical arc. **7.** A horizontal straight line. **8.** A concentric circle. **9.** A circle. **10.** A circle. **11.** The perpendicular bisector of BC. **13.** An arc of a circle. **14.** (i) A straight line, (ii) a circle. **15.** (1) A circle, (2) A straight line. **16.** An arc of a circle. **17.** A straight line. **18.** A straight line parallel to the base. **19.** A concentric circle. **20.** The produced parts of the sides of a rectangle. **23.** Find R such that AB bisects $\angle PRQ$. **24.** Two points. **26.** Four points.

REVISION PAPERS.

Paper I.

1. $A = 120°$, $B = 48°$, $C = 12°$. 3. Tetrahedron.

Paper II.

1. No. 3. If the axes of symmetry be not at right-angles, there would be more than two such axes.

Paper III.

3. 261 yds. S. E.

Paper V.

4. No.

Paper VI.

1. The median, the bisector and the altitude coincide. 2. A cone. A circle. 3. $(n - 2)$ triangles.

Paper VII.

3. If two sides of a triangle are unequal, the greater median bisects the lesser side and *vice-versa*.

Paper VIII.

2. Produce AC to B′ making CB′ = CA. With centres C, B′ draw arcs of radii CA, AB respectively to cut at A′. Join AA′ and let it cut BC at P. Then AP = 2 PC. 3. Yes, if $\angle$B or $\angle$C is obtuse. 4. Two triangles.

Paper IX.

1. At C make $\angle ECA = \angle BAC$ so that E lies in AB. Then $\angle B > \angle E$, *i. e.*, $> 2\angle A$. 4. Prove that $\angle BAC = 50°$.

Paper X.

2. Apply $\triangle$ DEF to $\triangle$ ABC so that D may fall on B and E on A and F at F′ on the side of AB opposite to C. Join CF′ and compare triangles F′BC, CAF′.

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vide *Fort St. George Gazette,* Part I-B, p. 545, dated 1-11-'27

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